- That is, q defines the "spacing" of ρ_F energy levels, and this spacing must be very small in order to be able to cancel an arbitrary value of $\Lambda<0$ to the desired precision.

- However, in string theory, there could be a very large number J of 4-form fields, e.g. $J \simeq 100-500$. In that case,

$$\rho_{vac} = \Lambda m_{pe}^2 + \frac{1}{2} \sum_{i=1}^{J} n_i^2 q_i^2 m_{pe}^4$$

For simplicity, assume all the fields have the same charge, $q_i = q$. Then each combination $\{n_1, n_2, \ldots, n_J\}$ describes a different vacuum, and the number of different vacuum (energies) is very large. The number of vac. states with $n^2 = \sum_{i=1}^{J} n_i^2$ in the range (n^2, n^2+dn^2) is

$$\left(\frac{dN}{dn^2}\right) dn^2 = \underbrace{\frac{(2\pi)^{J/2} \, n^{J-2}}{2\Gamma(J/2)}}_{\text{density of states} \,\propto\, \text{area of } J\text{-sphere of radius } n} dn^2$$

Spacing between states is now

$$4\pi q^2 m_{pe}^2 \Delta(n^2) \quad \text{where} \quad \Delta(n^2) = \frac{1}{dN/dn^2}$$

- For $\Lambda \approx \frac{m_{pe}^2}{4\pi}$, we want $n^2 q^2 m_{pe}^4 \sim \frac{m_{pe}^4}{4\pi} \Rightarrow n^2 \sim \frac{1}{4\pi q^2}$

- For $4\pi q^2 \sim 10^{-2}$ (like electric charge), we find $\Delta(n^2) \simeq 10^{-120}$ if $J \simeq 200$. $\therefore$ Presence of many 4-form fields $\Rightarrow$ closely spaced levels

- ∴ should be able to find a vacuum state w/ $\rho_{vac} \simeq 10^{-120} m_{pl}^4$ out of this very large ensemble of vacuum states.

- Then need to ask why we live in the Universe w/ this value of ρ_{vac} and not one of the others, in particular not one of the ones w/ $\rho_{vac} \gg 10^{-120} m_{pl}^4$.

- The favored answer appears to be the Anthropic Principle:

our existence as observers is a selection principle among this very large # of string vacuum states; The vacuum energy ~~Weinberg (1987)~~ must take on values that allow for the formation of observers ⇒ formation of galaxies + stars.

Cf. Weinberg (1987, 1989).

To understand the argument, first need to consider basics of Structure Formation:

- consider a Universe of collisionless (pressureless), non-relativistic particles (CDM) and dark energy that does not cluster.
- consider perturbations in the matter density: $\rho_m(\vec{x},t) = \bar{\rho}_m(t)\left[1+\delta_m(\vec{x},t)\right]$ at late times and with characteristic wavelengths $\lambda \ll H_0^{-1}$: can describe via Newtonian theory.

Newtonian Perturbation Theory for LSS :

e.g. Peebles, The LSS of the Universe (1980)

Fluid equations: in physical coordinates $\vec{r}$

$$\left.\frac{\partial \rho_m}{\partial t}\right)_{\vec{r}} + \vec{\nabla}_r \cdot \rho_m \vec{u} = 0$$

mass conservation

$$\rho_m\left[\left(\frac{\partial \vec{u}}{\partial t}\right)_{\vec{r}} + (\vec{u}\cdot\vec{\nabla}_r)\vec{u}\right] = -\rho_m \vec{\nabla}_r \Phi$$

Momentum conservation

Transform to comoving coordinates:

$$\vec{x} = \vec{r}/a(t) \qquad \vec{r} = a(t)\,\vec{x}$$

Physical velocity : $\vec{u} = \frac{d\vec{r}}{dt} = \underbrace{a\frac{d\vec{x}}{dt}} + \underbrace{\vec{x}\frac{da}{dt}} = \vec{v}(\vec{x},t) + \frac{\dot{a}}{a}\vec{r}$

Hubble flow : $\vec{x} = $ const.

peculiar velocity wrt Hubble flow $\equiv \vec{v}(\vec{x},t)$

Gravitational Potential : $\phi = \Phi + \frac{1}{2}a\ddot{a}x^2$

$$\nabla^2\phi = 4\pi G a^2 \bar{\rho}_m(t)\,\delta$$ sourced by perturbations

In comoving coords., fluid EOM becomes

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{a}(\vec{v}\cdot\vec{\nabla})_x \vec{v} + H\vec{v} = -\frac{1}{a}\vec{\nabla}_x\phi$$

$$\frac{\partial \delta_m}{\partial t} + \frac{1}{a}\vec{\nabla}_x \cdot (1+\delta_m)\vec{v} = 0$$

multiply 1st equ. by ρ, 2nd by $\vec{v}$, add them, and take divergence:

$$\ddot{\delta}_m + 2H\dot{\delta}_m = \frac{1}{a^2}\vec{\nabla}\cdot(1+\delta_m)\vec{\nabla}\phi + \frac{1}{a^2}\partial_\alpha\partial_\beta\left[(1+\delta_m)v^\alpha v^\beta\right]$$

Linear Perturbations in CDM:

- On large scales d, perturbation amplitude δ_m is small $\Rightarrow$ linearize:

$$\delta_m \ll 1 \,, \quad \left(\frac{vt}{d}\right)^2 \ll \delta_m$$

small velocities

$$\Rightarrow \boxed{\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G \bar{\rho}_m \delta_m}$$

$$4\pi G \bar{\rho}_m(t) = 4\pi G\,\bar{\rho}(t)\,\Omega_m(t) = \frac{3}{2}H^2\Omega_m$$

$$\cancel{H^2 = \frac{8\pi G}{3}\left(\bar{\rho}_m(t) + \rho_{DE}(t)\right)}$$

$$\cancel{\frac{3}{2}H^2 = 4\pi G\bar{\rho}\left[\Omega_m(t) + \Omega_{DE}(t)\right]}$$

Can also write this as

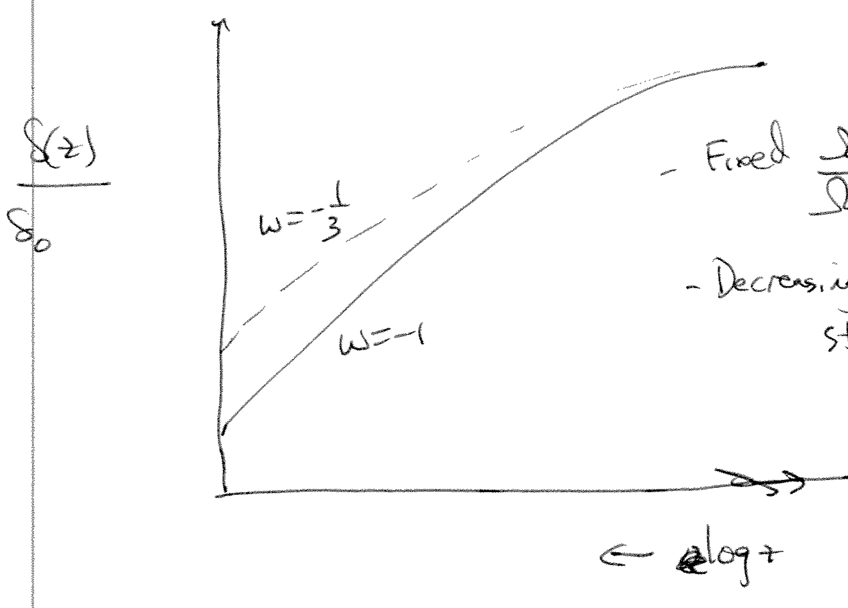
$$\boxed{\ddot{\delta}_m + 2H\dot{\delta}_m - \frac{3}{2}\Omega_m(t)H^2(t)\delta_m = 0}$$

"Friction" due to expanding Universe: timescale H^{-1}

Driving force with timescale $t \sim \frac{1}{\sqrt{\Omega_m}}\frac{1}{H}$

- At early times, $\bar{\rho}_m \gg \rho_{DE}$ since $\bar{\rho}_m \sim a^{-3}$ while $\rho_{DE} \sim a^{-3(1+w)}$, $\therefore$ at early times $\Omega_m(t) = 1$ and there is a growing mode solution with $\underline{\delta_m(\vec{x},t) \propto a(t)}$.

- When DE starts to dominate, $\Omega_m(t)$ drops below 1, damping term wins over gravitational driving term, perturbation growth stops: $\delta_m \to$ const. Pert. growth effectively stops at redshift $1+z = \left(\frac{\Omega_{m,0}}{\Omega_{DE,0}}\right)^{1/3w}$



- Fixed $\frac{\Omega_m}{\Omega_{DE}}$

- Decreasing Ω_m at fixed w also stops growth earlier.

- For increasing w, $\rho_{DE}(a)$ scales more rapidly with a $\Rightarrow$ dominates earlier for fixed Ω_m/Ω_{DE} $\Rightarrow$ pert. growth stops sooner.

- For $w_{DE} = -1$, have an exact growing mode soln:

$$\delta_m(z) \propto H(z)\, \frac{5\Omega_m}{2} \int_z^\infty \frac{1+z}{(H(z))^3}\, dz$$

$$\delta_m(a) \propto \frac{5}{2} H_0^2 \Omega_m H(a) \int_0^a \frac{da}{(a H(a))^3}$$

Return to Anthropic Principle:

- Overdense regions (mildly with $\delta < 1$) will not collapse ($\delta \to 1$) to form galaxies once DE comes to dominate.

- Let's assume that for galaxies to form massive stars which evolve to create heavy elements necessary for life, they must have formed at least as early as $z_f \sim 4$

(we see galaxies at higher redshift, of course, but we're not requiring them to have formed).

- For DE $= \Lambda$, this crudely implies

$$1 + z_f \gtrsim \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{-1/3}$$

$$\frac{\Omega_\Lambda}{\Omega_m} \leq (1+z_f)^3 \sim 125$$

Anthropic upper bound on Ω_Λ.

Note, observationally, $\Omega_\Lambda \simeq 3\,\Omega_m$.

- Note, in this argument, we have implicitly held fixed the amplitude of primordial density perturbations, which is presumably fixed by inflation, and it's quite plausible that would vary from string vacuum to string vacuum $\Rightarrow$ would loosen this limit: we could increase $\delta_{initial}$ to a larger value s.t. δ_{rms}(galaxy scale) ~ 1 when larger Λ starts to dominate. Then have to ask how many observers (probabilistically) you're requiring.

- This argument only gives us an upper bound on Λ that is ~ 2 orders of magnitude larger than the observed value, and allowing S_i to vary would lead to an even weaker upper bound.

- People have tried to strengthen this bound (Efstathiou) (Martel, Shapiro, Weinberg) based on probability arguments: a Universe w/ $\Omega_\Lambda = 3\Omega_m$ will have many more galaxies $\therefore$ more observers than one with $\Omega_\Lambda \simeq 100\Omega_m$, because structure formation proceeds for longer. $\therefore$ "Most" observers will measure a value for Ω_Λ smaller than the upper bound above:

$$\underbrace{dP(\rho_\Lambda)}_{\text{probability for measuring } \rho_\Lambda} = \underbrace{\nu(\rho_\Lambda)}_{\text{avg. \# of galaxies that form at specified value of } \rho_\Lambda \text{ (again, depends on } S_{\text{initial}})} \underbrace{P_*(\rho_\Lambda)\,d\rho_\Lambda}_{\text{A priori probability}}$$

probability for measuring ρ_Λ

avg. # of galaxies that form at specified value of ρ_Λ (again, depends on S_{initial})

A priori probability: assumed flat over the range of interest, but could be strongly peaked (Garriga + Vilenkin)

- Generally find that $\Omega_\Lambda \simeq 3\Omega_m$ is at least "reasonably likely".