- Coordinate distance from origin to FO at redshift z_e is then

$$\chi_e = \frac{c}{H_0}\int_0^{z_e} \frac{dz}{E(z)} = \int_{\frac{1}{1+z}}^{1} \frac{da}{a^2 H(a)}$$

where $E(z) = \frac{H(z)}{H_0} = \left[\Omega_m (1+z)^3 + \underbrace{(1-\Omega_m-\Omega_\Lambda)}_{\text{spatial curvature}}(1+z)^2 + \Omega_{DE}(1+z)^{3(1+w)}\right]^{1/2}$

Dark Energy, w/ $w=$const., reduces to $\Omega_\Lambda (1+z)^0 = \Omega_\Lambda$ for $w=-1$. More generally,

$$= \Omega_{DE} \exp\left[3\int (1+w(z'))\, d\ln(1+z')\right]$$

for non-const. $w(z)$

lecture 4 end

lecture 5:

- Then the "effective distance" is

$$r_e = S_k(\chi_e) = \frac{1}{|\Omega_k|^{1/2}} S_k\left[\frac{|\Omega_k|^{1/2} c}{H_0}\int_0^{z_e}\frac{dz}{E(z)}\right]$$

where $\Omega_k = 1-\Omega_m-\Omega_{DE}$

- Special case: matter-dominated, $\Lambda=0$, no Dark Energy:

$$r_e = \frac{2c\left[\Omega_m z_e + (\Omega_m - 2)\left\{\sqrt{1+\Omega_m z_e} - 1\right\}\right]}{H_0 \Omega_m^2 (1+z_e)}$$

Mattig 1958

$$= \frac{c}{H_0}\frac{z_e}{(1+z_e)}\frac{\left(1+\sqrt{1+\Omega_m z_e} + z_e\right)}{\left(1+\sqrt{1+\Omega_m z_e} + \frac{\Omega_m z_e}{2}\right)}$$

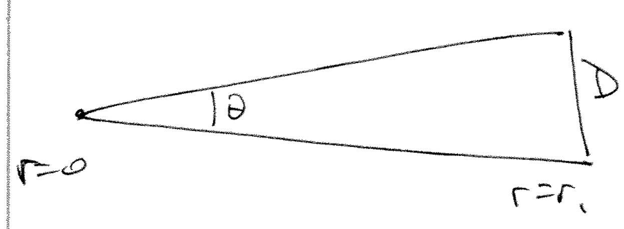
Expand in powers of z:

$$r = \frac{c}{H_0}\left[z - \frac{1}{2}z^2(1+q_0) + \dots\right] \quad \text{for } z \ll 1$$

where, as before, $q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_0$

See Fig. 2 of Rio lectures for examples

Angular Diameter Distance:



Observer at $r=0$, t_0 sees source of proper diameter D at coord. dist. r_1, which emitted light at $t=t_1$. Imagine source is oriented along the θ-axis.

- Proper length: imagine army of observers who can lay rulers end to end instantaneously (i.e. simultaneously).

- Proper size of the source is given by integrating ds along θ direction at fixed time: $ds = a(t_1)\, r_1\, d\theta$

$$\Rightarrow \quad D = a(t_1)\, r_1\, \theta$$

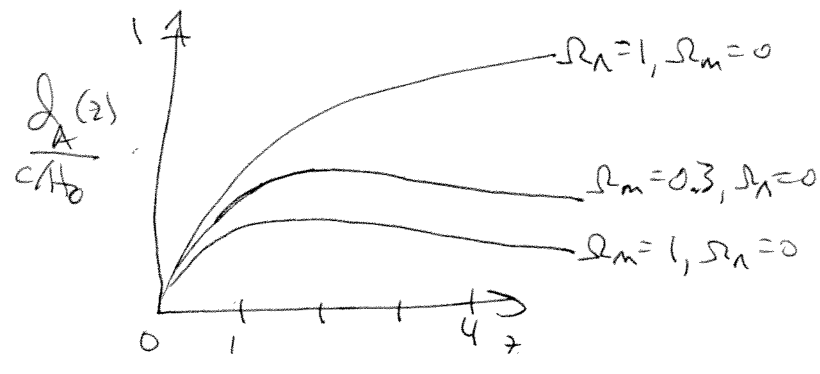
- Angular diameter of the source is then $\theta = \dfrac{D}{a(t_1)\, r_1}$

- In Euclidean geometry, $d = \dfrac{D}{\theta}$, so we <u>define</u> <u>angular diameter distance</u> to obey the same relation:

stick in $|\Omega_k|^{1/2}$ factors

$$\boxed{d_A \equiv \frac{D}{\theta} = a(t_1)\, r_1 = a(t_1)\, S_k(\chi_1) = \frac{r_1}{1+z_1} = \frac{S_k(\chi_1)}{1+z_1}}$$

See Peacock Fig. 3.7. for examples



- To implement this test, need calibrated (standard) rulers. e.g., the BAO feature.

BAO + Distances:

- In principle can measure LSS $\perp$ to LOS $\Rightarrow d_A(z)$
  and " $\parallel$ " " $\Rightarrow H(z)$

if we have a standard ruler, such as a fixed physical lengthscale expected in the clustering. For BAO/CMB, this is the sound horizon distance at epoch of CMB last scattering: $s = c_s t_{LS}$ ⊛. Then $\xi_{gg}(r)$ shows a feature at a scale related to s.

- Current data: controversy over whether $\perp$ and $\parallel$ clustering can be separated, i.e. use spherical avg. of clustering $\Rightarrow$

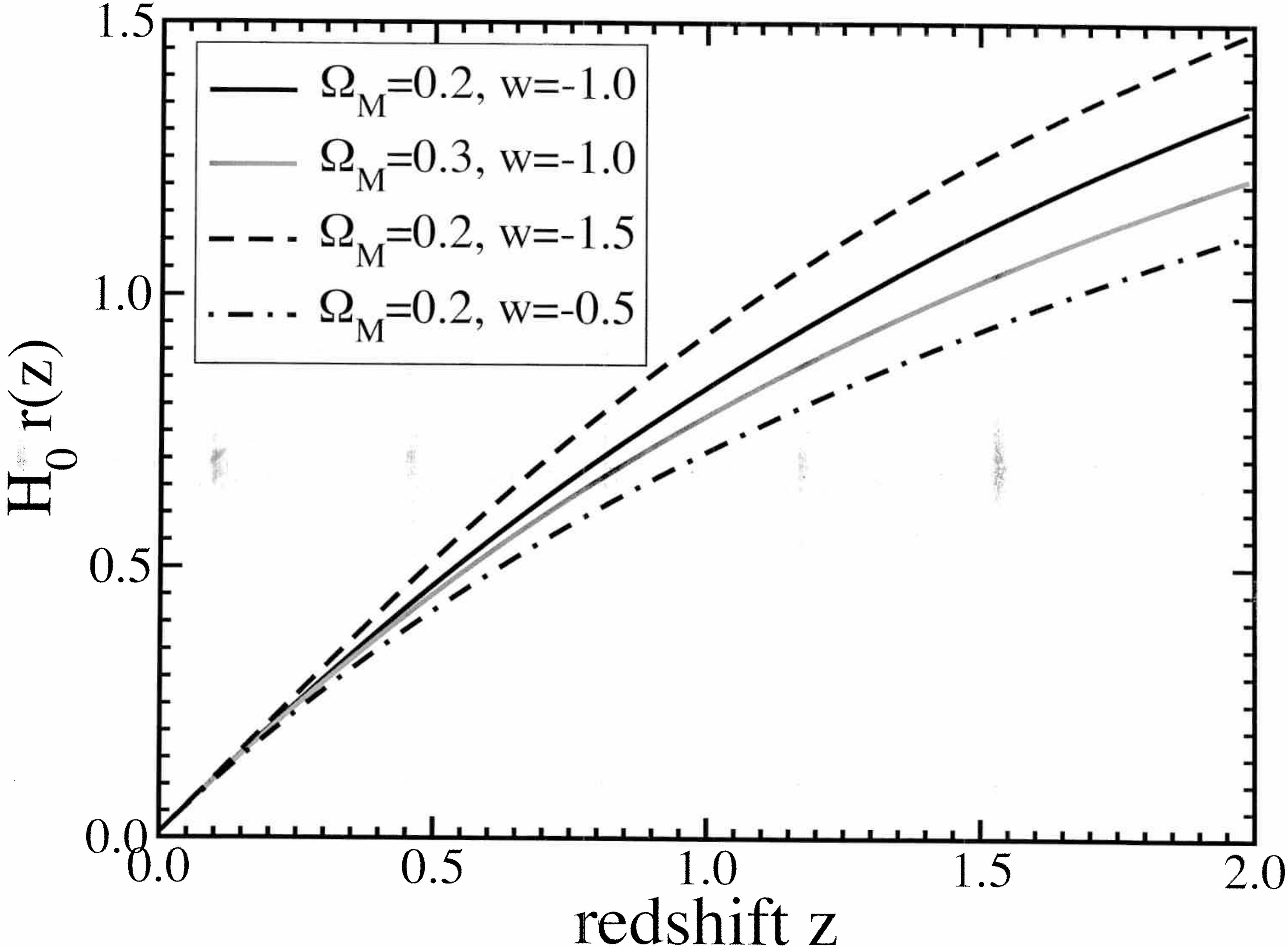
$$D_V(z) = \left[(1+z)^2 d_A^2(z) \frac{cz}{H(z)} \right]^{1/3}$$

⊛ Comoving sound-horizon distance is

$$r_s(z) = \frac{c}{\sqrt{3}} \int_0^{\frac{1}{1+z}} \frac{da}{a^2 H(a) \left[1 + \underbrace{\left(\frac{3\Omega_b}{\Omega_\gamma}\right) a}_{c_s} \right]^{1/2}}$$

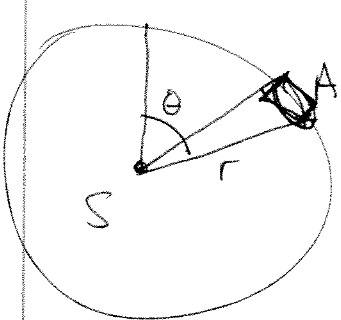
Then BAO feature determined by $r_s(z_{dec}) / D_V(z_{gal})$

- CMB sensitive to: $\dfrac{d_A(z_{LS})}{r_s(z_{LS})}$ and $\dfrac{d_A(z_{LS})}{(c/H(z_{LS}))}$



$H_0\ r(z)$
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
redshift z
Ω_M=0.2, w=-1.0
Ω_M=0.3, w=-1.0
Ω_M=0.2, w=-1.5
Ω_M=0.2, w=-0.5

Luminosity Distance



- Source S at origin emits light at time t_1 into solid angle $d\Omega$ which is received by O at coord. dist. r at time t_0, with detector of area A. (rectangular, in the θ, φ directions)

- The proper area A of the detector is given by using the proper distance,

$$dl_{proper} = \left(-ds^2\right)^{1/2}\Big|_{dt=0} \Rightarrow$$

$$A = a(t_0)\, r\, d\theta\; a(t_0)\, r \sin\theta\, d\varphi = a_0^2 r^2 d\Omega$$

where $d\Omega = \sin\theta\, d\theta\, d\varphi$ = element of solid angle

∴ A unit area at O subtends solid angle $d\Omega = \frac{1}{a_0^2 r^2}$ at S.

- Power (energy/time) emitted into $d\Omega$ is

$$dP = \frac{L\, d\Omega}{4\pi} \quad \text{where } L = \text{luminosity of } S$$

- Energy flux received by O per unit area, in absence of expansion, is

$$f = \frac{L\, d\Omega}{4\pi} = \frac{L}{4\pi a_0^2 r^2}$$

- Must include the effects of expansion:

1.) Photon energy redshifts: $E_\gamma(t_0) = E_\gamma(t_1)/(1+z)$

2.) Photons emitted at time intervals δt_e arrive more slowly, at intervals $\delta t_0 = \delta t_e \frac{a_0}{a(t_1)} = \delta t_e/(1+z)$

- Thus, received energy flux is

$$f = \frac{L}{4\pi a_0^2 r^2 (1+z)^2}$$

- Define <u>luminosity distance</u> as that corresponding to the inverse square law: $d_L^2 = \frac{L}{4\pi f}$

$$\Rightarrow \boxed{d_L = a_0 r (1+z) = r(1+z) = d_A (1+z)^2}$$

$$= c(1+z)\, S_k\left[\frac{1}{H_0}\int \frac{da}{a^2 E(a)}\right]$$

- For example, for NR matter + Λ, we have

$$d_L(z; \underset{H_0}{\Omega_m}, \Omega_\Lambda) = c(1+z)\, S_k\left[\int \frac{da}{H_0 a^2 \left[\Omega_m a^{-3} + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda) a^{-2}\right]^{+1/2}}\right]$$

- Note: $H_0 d_L(z; \Omega_m, \Omega_\Lambda)$ is indep. of H_0.

<u>- Distance Modulus:</u>

- Absolute magnitude: $M = -2.5\log_{10}(L) + \text{const}_1$
- Apparent magnitude: $m = -2.5\log f + \text{const}_2$.
- Distance modulus:

$$\mu = m - M = 2.5\log_{10}(L/f) + c_3$$

$$= 5\log_{10}(d_L/10\,\text{pc})$$

$$= 5\log_{10}(H_0 d_L(z; \Omega_m, \Omega_\Lambda)) - 5\log_{10} H_0 + \text{const.}$$

- Complications: K-corrections, Extinction

Assume this is fixed: standard candles.

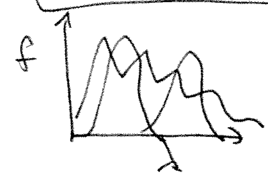
$$\mu = m_X - M_Y = 5\log_{10}(H_0 d_L) - 5\log_{10} H_0 + K_{XY} + A_i$$

K-correction: observer passband captures different fraction of source light than the emitted passband

extinction

X = observer passband

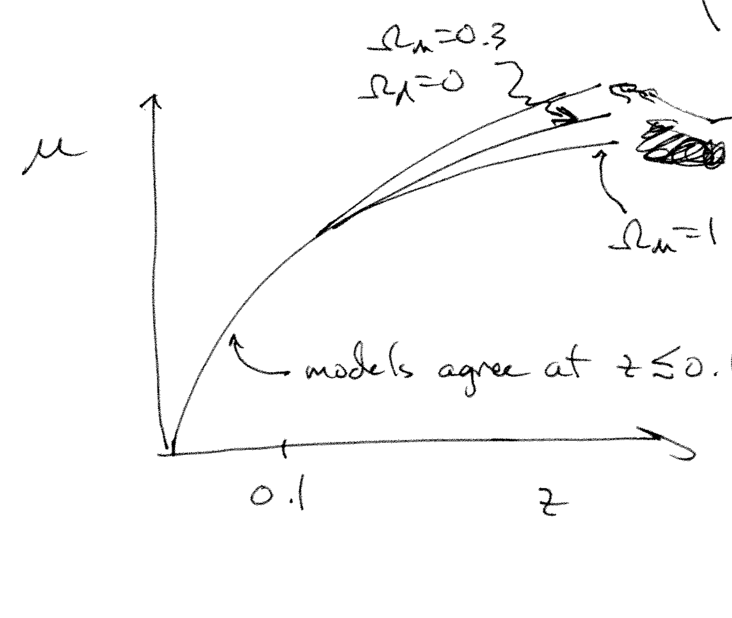
Y = rest-frame passband (in which the object is a standard candle.)



- Generally, we don't have absolute calibration of M_Y nor precise knowledge of H_0 $\Rightarrow$ only measure relative distances at two different redshifts, z_1 and z_2:

$$\mu_1 - \mu_2 = m_{X,1} - M_Y - m_{X,2} + M_Y = 5\log_{10}\left(\frac{d_1}{d_2}\right) + K_1 + A_1 - K_2 - A_2$$

indep. of M or H_0

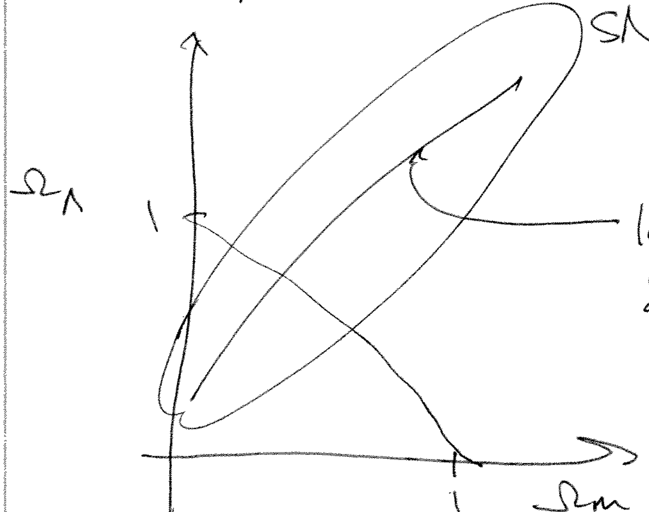


: at low z, $\mu \cong 5\log v - 5\log H_0$, reflecting $v \cong H_0 d$.

Shifting M, H_0 shifts $\mu(z)$ vertically by const. amount, indep. of z, so no substantial degeneracy w/ cosmological parameters.

- Type Ia supernovae: relative distances det'd to ~7%, i.e., show scatter of ~0.15 mag around best-fit Hubble law model.

- measure distance moduli for ~few hundred SNe Ia at low ($z \lesssim 0.1$) and high ($z \sim 0.5-1$) redshift these days.



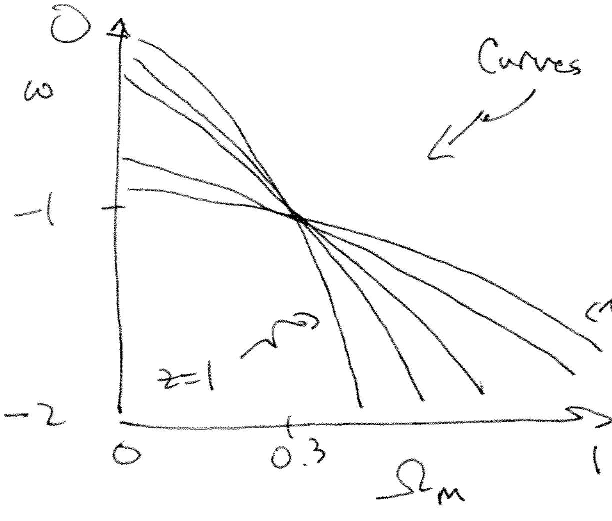
See Fig. 3 of Ris lectures: discovery data

Ris slides 57–67

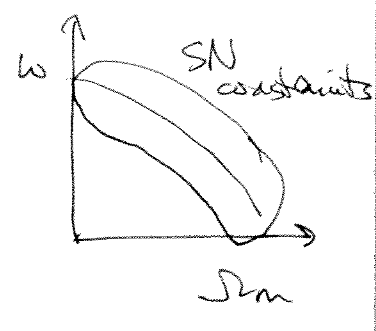
locus of constant μ at $z \simeq 0.5 \Rightarrow$ defines d_L-degeneracy in the $\Omega_\Lambda - \Omega_m$ plane. With increasing z, this locus rotates slowly in the plane.

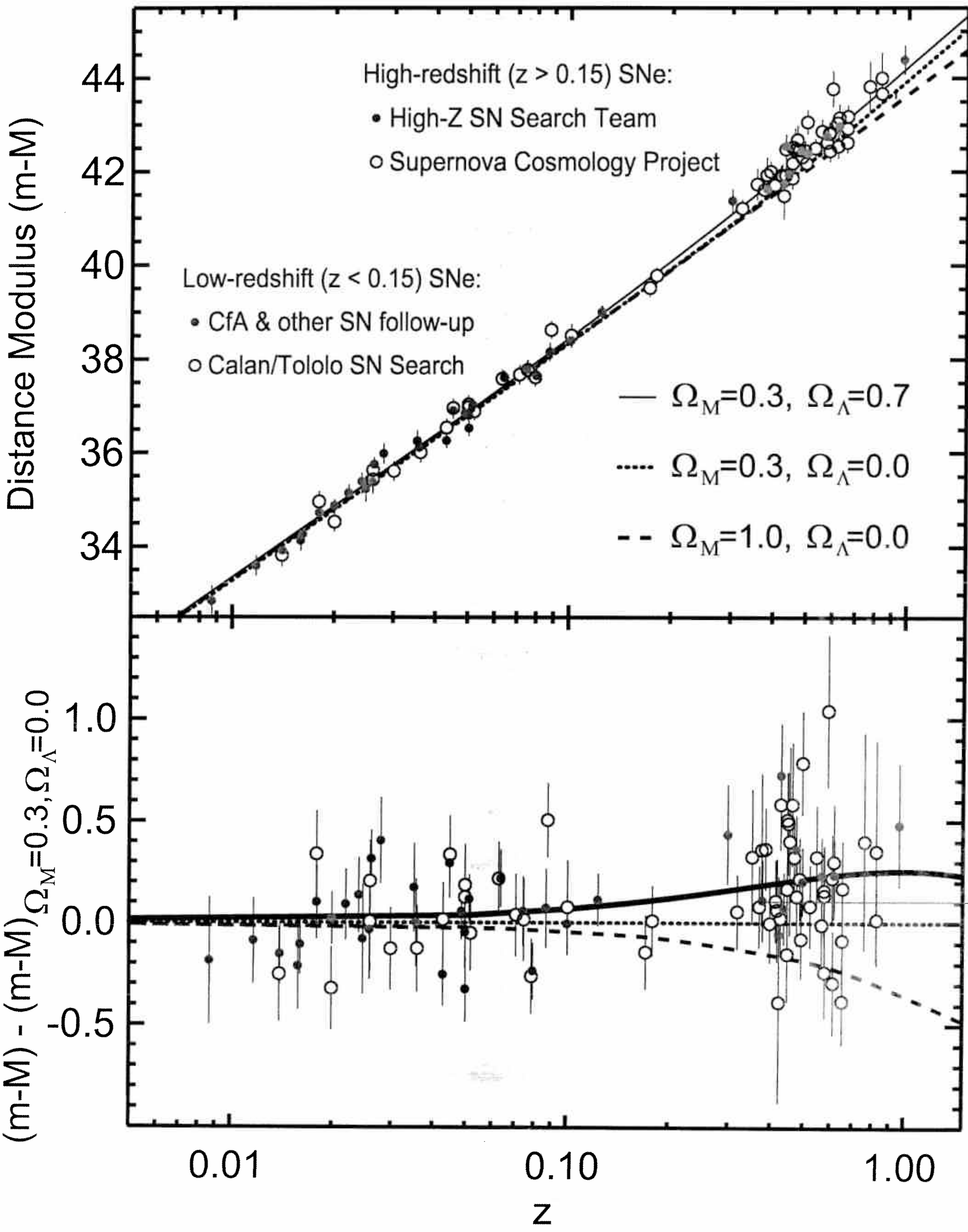
- Constraints above assume DE $= \Lambda$, with $w = -1$.
- Instead can constrain another class of models: $k = 0$ ($\Omega_m + \Omega_{DE} = 1$) with const. w

Curves of constant d_L at fixed z



$\Rightarrow$





High-redshift (z > 0.15) SNe:
High-Z SN Search Team
Supernova Cosmology Project
Low-redshift (z < 0.15) SNe:
CfA & other SN follow-up
Calan/Tololo SN Search
Ω_M=0.3, Ω_Λ=0.7
Ω_M=0.3, Ω_Λ=0.0
Ω_M=1.0, Ω_Λ=0.0
Distance Modulus (m-M)
44
42
40
38
36
34
$(m\text{-}M) - (m\text{-}M)_{\Omega_M=0.3,\Omega_\Lambda=0.0}$
1.0
0.5
0.0
-0.5
0.01
0.10
1.00
z

Volume Element

- If we have a set of objects (e.g., galaxies, clusters, dark matter halos) whose spatial number density we know / can predict, then counting them provides another cosmological test. We'll apply this espec. to galaxy cluster counts.

- ~~Recall that~~ A proper area dA at redshift z and radial coordinate r subtends solid angle $d\Omega$ at the origin given by

$$dA = a(t_e)\, r\, d\theta\, a(t_e)\, r \sin\theta\, d\phi$$
$$= a_e^2 r^2 d\Omega = \frac{a_0^2 r^2 d\Omega}{(1+z)^2}$$

- The rate of proper displacement w/ z along a light ray is

$$dl = c\,dt = \frac{da}{\dot{a}} = \frac{dz}{(1+z)} \frac{a}{\dot{a}} = \frac{dz}{H(z)(1+z)}$$

= linear depth of a sample of objects in the redshift interval $(z, z+dz)$

- Proper volume element of the sample is then

$$d^2V_{prop} = dA\, dl = \frac{a_0^2 r^2(z)}{H_0 E(z)(1+z)^3}\, d\Omega\, dz$$

- If $n_{prop}(z)$ is the proper number density of objects, then the #counts per unit redshift and solid angle are

$$\boxed{\frac{d^2N}{dz\,d\Omega} = n_{prop}(z) \frac{d^2V_{prop}}{dz\,d\Omega} = \frac{n_{prop}(z)\, a_0^2 r^2(z)}{H_0 E(z)(1+z)^3}}$$

- If objects are conserved (neither created nor destroyed), then

$$n_{prop}(z) = n_0 (1+z)^3 \quad \Rightarrow \quad \frac{d^2N}{dz\,d\Omega} = \frac{n_0 \, r^2(z)}{H(z)}$$

- Convenient to define the comoving # density,

$$n_c(z) = \frac{n_{prop}(z)}{(1+z)^3} = \text{constant if objects are conserved}$$

$$= \text{\# density per comoving volume},$$

where the comoving volume element is defined by

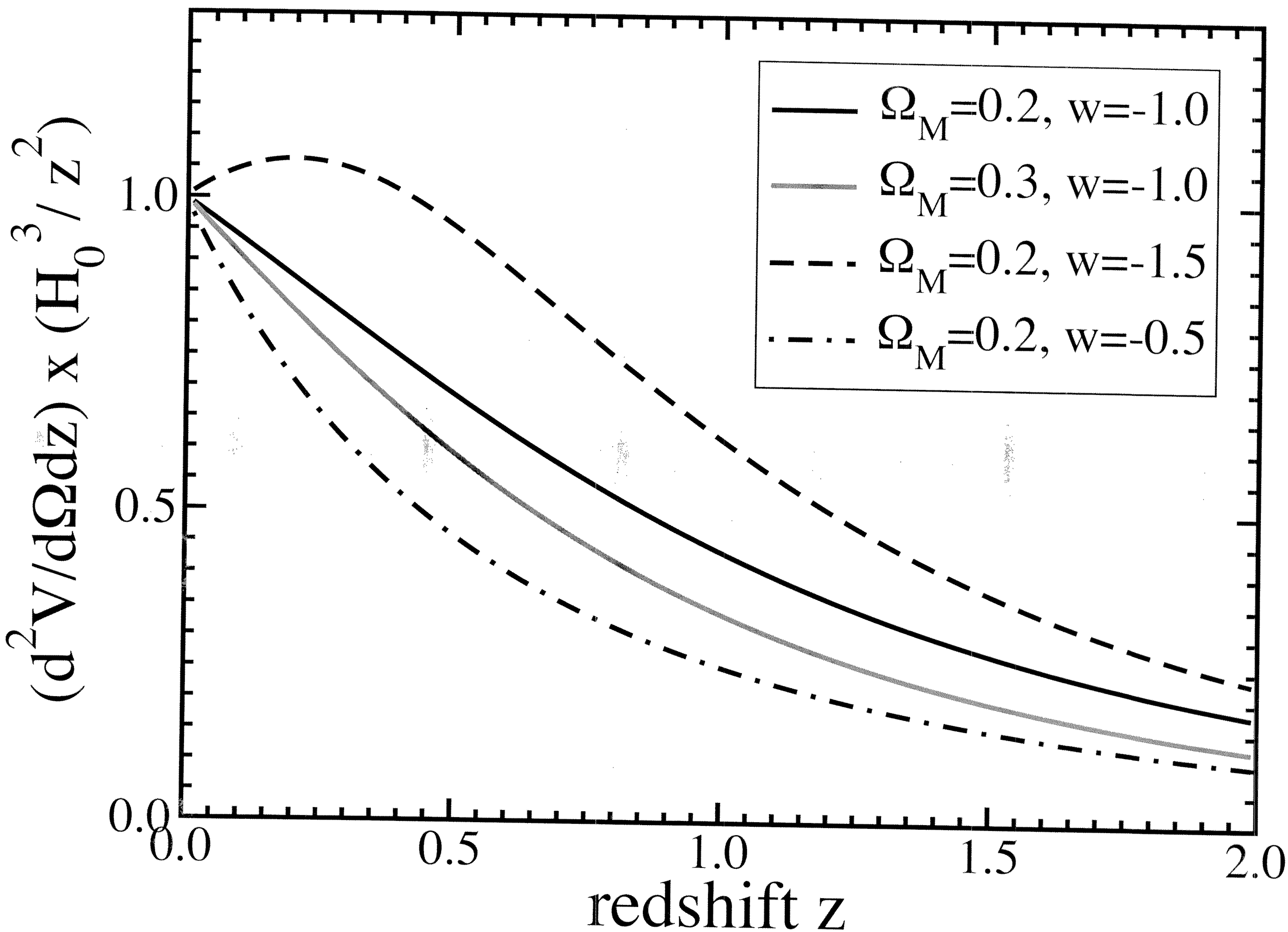
$$\frac{d^2V_c}{dz\,d\Omega} = \frac{r^2(z)}{H(z)} = \frac{r^2(z)}{H_0 E(z)} = \left(\frac{d^2V_{prop}}{dz\,d\Omega}\right)(1+z)^3$$

so that the counts are also given by

$$\boxed{\frac{d^2N}{dz\,d\Omega} = n_c(z) \frac{d^2V_c}{dz\,d\Omega} = \frac{n_c(z)\, r^2(z)}{H(z)}}$$

- See Fig. 2 right-hand panel of Rio lectures for examples

- We will apply this to clusters, for which theory predicts $n(z)$.



Ω_M=0.2, w=-1.0
Ω_M=0.3, w=-1.0
Ω_M=0.2, w=-1.5
Ω_M=0.2, w=-0.5
(d²V/dΩdz) x (H₀³/ z²)
0.0
0.5
1.0
0.0
0.5
1.0
1.5
2.0
redshift z

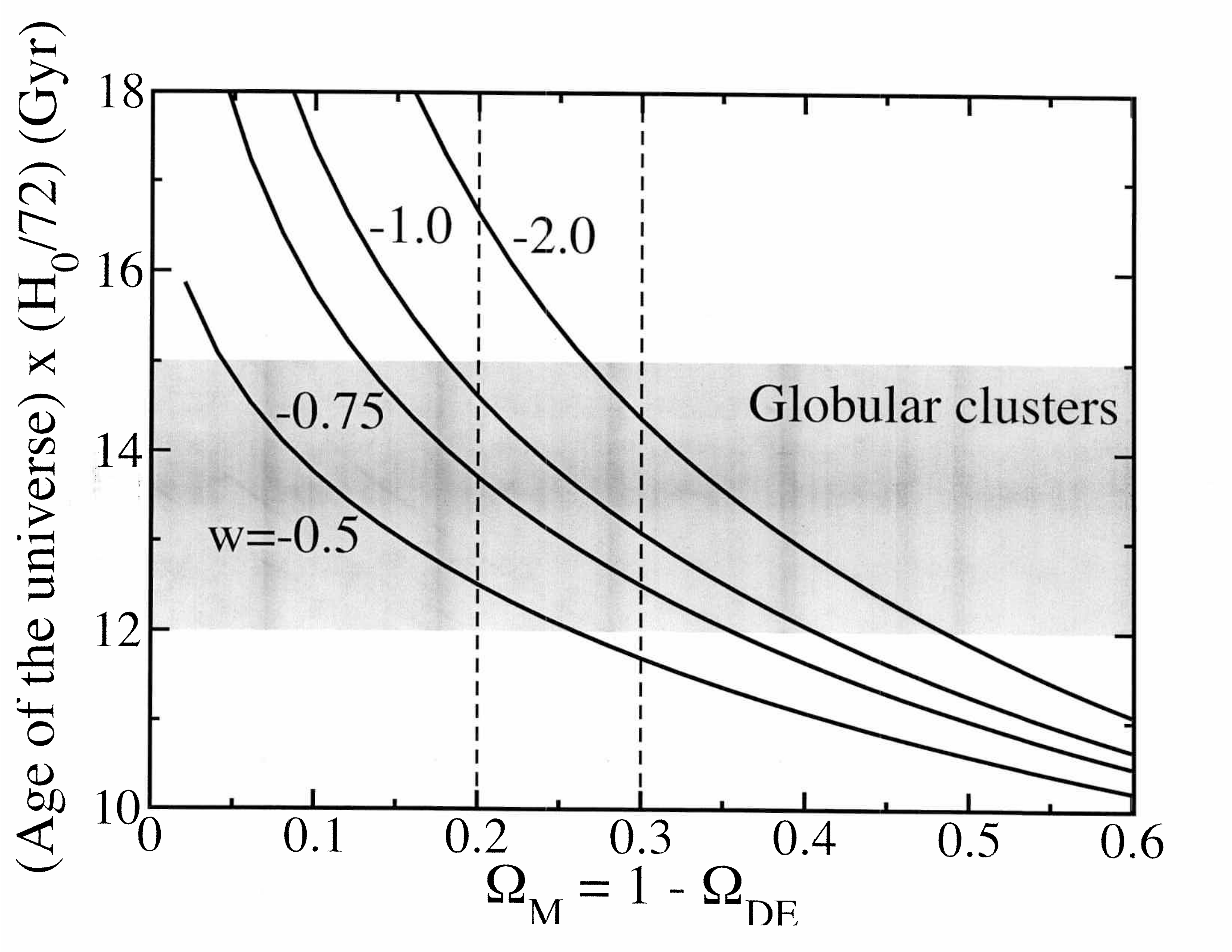
Age of the Universe:

- For completeness, the age of the Universe $t(a)$ when it had relative size $a(t)/a_0 = \frac{1}{1+z}$, is given by the same kind of integral over $H^{-1}(z)$:

$$t(z) = \int_0^{t(z)} dt' = \int_z^{\infty} \frac{dz'}{(1+z')\,H(z')}$$

and current age of the Universe is given by setting $z=0$.

- See t_0 vs. Ω_m plot for $k=0$ from Frieman, Turner, Huterer.


(Age of the universe) x (H0/72) (Gyr)
18
16
14
12
10
0
0.1
0.2
0.3
0.4
0.5
0.6
ΩM = 1 - ΩDE
-1.0
-2.0
-0.75
w=-0.5
Globular clusters