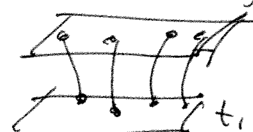
Lecture 4:

Relativistic Cosmology: Friedmann-Robertson Walker Model

- Homogeneity + isotropy $\Rightarrow$ can construct family of fundamental observers (FO's), locally at rest wrt the cosmic fluid, who carry along comoving coordinates $x^i(t)$ with them. Can foliate 4d spacetime into spacelike surfaces Σ_t of constant time and fluid density, which are orthogonal to the 4-velocities of the FO's.



- This implies that the spacetime metric can be written as

$$\boxed{g_{\mu\nu} = \mathrm{diag}(+1, -a^2(t)\, h_{ij})}$$

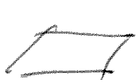
FRW scale factor — non-expanding 3-space metric, independent of t.

- Homogeneity + isotropy imply that the surfaces Σ_t (described by the metric h_{ij}) have spatially constant Riemann curvature, which means that

$$^{(3)}R_{abcd} = K(h_{ac}h_{bd} - h_{ad}h_{bc})$$

- There are only 3 3-spaces of constant curvature (assuming simple topology):

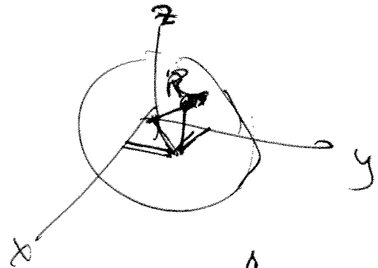
 $\underline{K>0}$: S^3, positive curvature, finite but no boundary

 $\underline{K=0}$: R^3, Euclidean, flat, infinite

 $\underline{K<0}$: H^3, 3-hyperboloid, negative curvature, infinite

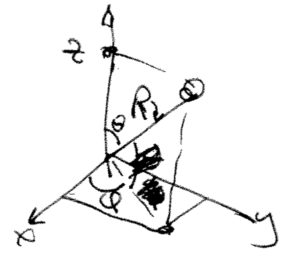
Metrics:

- First consider lower-dimensional example, S^2: this is the surface in Euclidean 3 dimensions given by



$$x^2+y^2+z^2=R^2$$

- Flat 3-d has metric



$$ds^2 = dx^2+dy^2+dz^2$$

- Spherical coordinates: $(x,y,z) \rightarrow (R,\theta,\varphi)$

$x = R\sin\theta\cos\varphi$, $y = R\sin\theta\sin\varphi$, $z = R\cos\theta$, $R = \text{const.}$

- Metric on S^2 is therefore $ds_2^2 = R^2(d\theta^2+\sin^2\theta\, d\varphi^2)$

- Generalize to S^3: in flat R^4, $ds^2 = dx^2+dy^2+dz^2+dw^2$

$(x,y,z,w) \rightarrow (R,\theta,\varphi,\chi)$ defined by

$x = R\sin\chi\sin\theta\cos\varphi$, $y = R\sin\chi\sin\theta\cos\varphi$, $z = R\sin\chi\cos\varphi$

$w = R\cos\chi \Rightarrow$

$$ds_3^2 = R^2\left[d\chi^2+\sin^2\chi\,(d\theta^2+\sin^2\theta\, d\varphi^2)\right]$$

- For R^3, H^3: same as above, but replace $\sin^2\chi$:

$$\boxed{ds_3^2 = d\chi^2 + S_k^2(\chi)\,(d\theta^2+\sin^2\theta\, d\varphi^2)}$$

$$S_k(\chi) = \begin{cases} \sin\chi & \text{for } k=+1 \quad S^3 \\ \chi & \text{for } k=0 \quad R^3 \\ \sinh\chi & \text{for } k=-1 \quad H^3 \end{cases} \qquad k = |K|/K$$

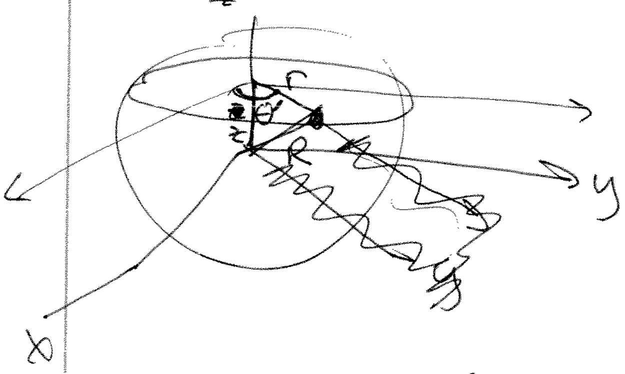
and $\boxed{ds^2 = dt^2 - a^2(t)\, ds_3^2}$

- Alternative form:

- Sometimes convenient to define a different radial coordinate, $r = S_k(\chi)$, in terms of which the metric becomes

$$\boxed{ds_3^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$

- Motivation for this form: again consider S^2, but now with spherical polar coordinates:



$x = r\cos\theta'$, $y = r\sin\theta'$, $z = \left[R^2 - r^2\right]^{1/2}$

Then $dx^2 + dy^2 = dr^2 + r^2 d\theta'^2$

$$dz^2 = \frac{r^2 dr^2}{(R^2 - r^2)}$$

$\therefore$ Metric on S_2^{r2} is

$$ds_2^2 = \left(dx^2 + dy^2 + dz^2\right)\Big|_{R=\text{const}} = r^2 d\theta'^2 + \left(1 + \frac{r^2}{(R^2 - r^2)}\right) dr^2$$

$$= r^2 d\theta'^2 + \frac{R^2 dr^2}{(R^2 - r^2)}$$

Now rescale: define $\tilde{r} = r/R$ $\quad (0 \leq \tilde{r} \leq 1) \Rightarrow$

$$ds_2^2 = R^2\left[\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2 d\theta'^2\right]$$

Note: coord. singularity at the poles, where $\tilde{r} \to 0$.

Conservation of Energy - Momentum:

- Einstein eqns. + Bianchi identities imply $D_\nu T^{\mu\nu} = 0$
- We'll model $T^{\mu\nu}$ as a perfect (multi-component) fluid,

so that $$T^{\mu\nu} = -pg^{\mu\nu} + (p+\rho)u^\mu u^\nu$$

where p = pressure , ρ = energy density , u^μ = fluid 4-velocity

and for perfect fluid we have set anisotropic stresses and viscosities to 0.

- In comoving frame (fluid rest frame), $u^0 = 1$, $u^i = 0$ $\Rightarrow$

$$T^\mu_\nu = \text{diag}(\rho, -p, -p, -p) \qquad g_{\mu\nu} = (+---)$$

- Can show that covariant divergence of 2nd rank tensor is given by
$$D_\nu T^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \left(\sqrt{g}\, T^{\mu\nu}\right) + \Gamma^\mu_{\nu\alpha} T^{\alpha\nu}$$

where $g \equiv -\det(g_{\mu\nu})$.

- Thus, $$D_\nu T^{\mu\nu} = \underbrace{D_\nu(-pg^{\mu\nu})}_{= -g^{\mu\nu}\partial_\nu p} + \frac{1}{\sqrt{g}} \partial_\nu \left(\sqrt{g}\,(p+\rho)u^\mu u^\nu\right) + \Gamma^\mu_{\beta\alpha}(p+\rho)u^\beta u^\alpha$$

$\mu=0$ component:

$$0 = \nabla_\nu T^{0\nu} = -\frac{\partial p}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left[\sqrt{g}\,(p+\rho)\right] + \Gamma^0_{00}(p+\rho)$$

For FRW metric:

$$g = -\det(g_{\mu\nu}) = a^6(t)\, f(\chi,\theta,\varphi)$$

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left\{\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right\} \Rightarrow$$

$$\Gamma^\alpha_{00} = \frac{1}{2} g^{\nu\alpha}\left\{\partial_0 g_{0\nu} + \partial_0 g_{0\nu} - \partial_\nu g_{00}\right\}$$

But $g_{0\nu}$ is only non-zero for $\nu=0$ (diagonal metric),

and $g_{00}=1 \Rightarrow \partial(g_{00})=0 \Rightarrow \underline{\Gamma^\alpha_{00}=0}$

$\therefore$ $\mu=0$ component becomes:

$$\frac{dp}{dt} - \frac{1}{a^3(t)}\frac{d}{dt}\left[a^3(t)(p+\rho)\right]=0$$

$$\Rightarrow \quad \dot{p} - (p+\rho)\,3\frac{\dot{a}}{a} - \dot{p} - \dot{\rho} = 0$$

$$\Rightarrow \quad \boxed{\dot{\rho} + 3H(p+\rho) = 0}$$

as we found from 1st law of Thermodynamics

FRW Dynamics:

- Need a few more components of $\Gamma^{\mu}_{\nu\alpha}$:

$$\Gamma^{0}_{ij} = \frac{1}{2} g^{00}\left(\underbrace{\partial_j g_{i0} + \partial_i g_{j0}}_{\text{vanish by symmetry of } g_{\mu\nu}} - \partial_0 g_{ij}\right) = -\frac{1}{2}\partial_t g_{ij}$$

where $g_{ij} = -a^2(t) h_{ij}(\vec{x})$

$$\Rightarrow \Gamma^{0}_{ij} = a\dot{a} h_{ij} = -\frac{\dot{a}}{a} g_{ij}$$

Also,

$$\Gamma^{i}_{0j} = \frac{1}{2} g^{ik}\left(\cancel{\partial_j g_{0k}} + \partial_0 g_{jk} - \cancel{\partial_k g_{0j}}\right)$$

$$= \frac{1}{2} g^{ik}\, 2\frac{\dot{a}}{a} g_{jk} = \frac{\dot{a}}{a}\delta^i_j \qquad \therefore \Gamma^{i}_{0i} = 3\frac{\dot{a}}{a}$$

Ricci tensor: $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}$

$$R_{00} = \cancel{\partial_\alpha \Gamma^{\alpha}_{00}} - \partial_0 \Gamma^{i}_{0i} + \cancel{\Gamma^{\alpha}_{00}\Gamma^{\beta}_{\alpha\beta}} - \Gamma^{i}_{0j}\Gamma^{j}_{0i}$$

$$= -3\partial_t\left(\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2 \underbrace{\delta^i_j \delta^j_i}_{=3} = -3\frac{\ddot{a}}{a}$$

Similarly, can show that

$$R_{ij} = -g_{ij}\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right]$$

so that

$$R = g^{\mu\nu} R_{\mu\nu} = -3\frac{\ddot{a}}{a} - 3\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

$$= -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

Field Eqns. for FRW:

0-0 component: $R_{00} - \frac{1}{2} g_{00} R - \Lambda g_{00} = 8\pi G T_{00}$

Substituting, $-\frac{3\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] - \Lambda = 8\pi G \rho$

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}}$$

ij component: $R_{ij} - \frac{1}{2} g_{ij} R - \Lambda g_{ij} = 8\pi G T_{ij} = -8\pi G g_{ij} p$

Substituting, $-g_{ij}\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right] + 3g_{ij}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] - \Lambda g_{ij} = -8\pi G p g_{ij}$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = -8\pi G p$$

Subtract 0-0 component: $\boxed{\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}}$

- As noted previously, we can absorb/rewrite the Λ terms using

$$p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}$$

- GR dynamics for $a(t)$ consistent w/ what we derived before.

Cosmological Distances:

- We will discuss a few measures of distance in cosmology that are relevant to observations.

Comoving Distance:

- For a radial null geodesic (light ray), $ds^2 = d\theta^2 = d\varphi^2 = 0$, so, from FRW metric, we have

$$c\,dt = a(t)\,d\chi \quad \text{(restoring speed of light explicitly)}$$

and the comoving distance χ is given by

$$\chi = \int \frac{c\,dt}{a(t)} = \int \frac{c\,dt}{a\,da}\,da = c\int \frac{da}{a^2 H(a)}$$

~~Without loss of generality, we can set~~

Redshift:

- Consider light ray emitted by FO_1 at $r = r_e$, $t = t_e$ reaching FO_2 at $r = 0$ at time t_0 and also assume radial, null geodesic path. Successive light waves emitted w/ spacing (period) δt_e reach FO_2 w/ spacing (between peaks of wave) δt_0. Derive the relation between them:

$$0 = ds^2 = dt^2 - a^2(t)\frac{dr^2}{1-kr^2} \quad \text{along the trajectory}$$

$$\Rightarrow \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^{r_e} \frac{dr}{(1-kr^2)^{1/2}} = \text{constant}, \text{ since } FO\text{'s remain at fixed/comoving coord. positions}$$

This implies $\displaystyle\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{dt}{a(t)}$ by constancy of comoving coord. separation

Rewrite this as

$$\int_{t_e}^{t_e+\delta t_e} \frac{dt}{a(t)} + \int_{t_e+\delta t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_o} \frac{dt}{a(t)} + \int_{t_o}^{t_o+\delta t_o} \frac{dt}{a(t)}$$

Assume $\delta t_e, \delta t_o \ll H_o^{-1} \simeq$ time over which $a(t)$ changes appreciably

$(\lambda \ll c H_o^{-1} : 10^3 \text{Å} \ll 10^{28}\,\text{cm}$ for optical light$)$

$$\Rightarrow \quad \frac{\delta t_e}{a(t_e)} = \frac{\delta t_o}{a(t_o)} \Rightarrow \underline{\frac{\delta t_o}{\delta t_e} = \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}}$$

- Redshift is <u>defined</u> by $z = \dfrac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$, $\Rightarrow$

$$\boxed{1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}}$$

Wavelength of light stretches w/ scale factor, again as noted before.

- Without loss of generality, we can always set $a(t_o) = a_o = 1$, so that $1+z = \dfrac{1}{a(t_e)}$ $\therefore a = \dfrac{1}{1+z} \Rightarrow da = \dfrac{-dz}{(1+z)^2} = -a^2 dz$

- Substitute this into earlier relation for χ:

$$\frac{c\,dt}{da} da = a\,d\chi \Rightarrow -\frac{c}{\dot{a}} a^2 dz = a\,d\chi \Rightarrow \boxed{-c\,dz = H(z)\,d\chi}$$

relation between redshift & comoving distance

$$\boxed{\chi = c \int \frac{dz}{H(z)}}$$

- Coordinate distance from origin to FO at redshift z_e is then

$$\chi_e = \frac{c}{H_0}\int_0^{z_e}\frac{dz}{E(z)} \quad \cancel{E(z)=\frac{H(z)}{H_0}} = \int_{\frac{1}{1+z}}^{1}\frac{da}{a^2 H(a)}$$

where $$E(z) = \frac{H(z)}{H_0} = \left[\Omega_m(1+z)^3 + (1-\Omega_m-\Omega_\Lambda)(1+z)^2 + \Omega_{DE}(1+z)^{3(1+w)}\right]^{1/2}$$

spatial curvature

Dark Energy, w/ $w=$const., reduces to $\Omega_\Lambda(1+z)^0 = \Omega_\Lambda$ for $w=-1$. More generally,

$$= \Omega_{DE}\exp\left[3\int(1+w(z'))\,d\ln(1+z')\right]$$

for non-const. $w(z)$

Lecture 4 end

- Then the "effective distance" is

$$r_e = S_k(\chi_e) = \frac{1}{|\Omega_k|^{1/2}} S_k\left[\frac{|\Omega_k|^{1/2}c}{H_0}\int_0^{z_e}\frac{dz}{E(z)}\right] \quad \text{where } \Omega_k = 1-\Omega_m-\Omega_{DE}$$

- Special case: matter-dominated, $\Lambda=0$, no Dark Energy:

$$r_e = \frac{2c\left[\Omega_m z_e + (\Omega_m-2)\left\{\sqrt{1+\Omega_m z_e}-1\right\}\right]}{H_0\Omega_m^2(1+z_e)} \quad \text{Mattig 1958}$$

$$= \frac{c}{H_0}\frac{z_e}{(1+z_e)}\frac{\left(1+\sqrt{1+\Omega_m z_e}+z_e\right)}{\left(1+\sqrt{1+\Omega_m z_e}+\frac{\Omega_m z_e}{2}\right)}$$

Expand in powers of z:

$$r_e = \frac{c}{H_0}\left[z - \frac{1}{2}z^2(1+q_0) + \dots\right] \quad \text{for } z \ll 1$$

where, as before, $q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_0$

See Fig. 2 of Rio lectures for examples