Lecture 12 : Modified Gravity (continued)

- f(R) Models:

Let's write $f(R) = R + g(R)$ and define

$$g_R \equiv \frac{\partial g}{\partial R}$$

Then the trace of the Einstein eqns. becomes

$$\Box g_R = \frac{1}{3}\left(R + 2g - R g_R\right) - \frac{8\pi G}{3}\left(\rho_m - 3p_m\right) \equiv \frac{\partial V_{eff}}{\partial g_R}$$

This is a 2nd order eqn. for g_R w/ canonical kinetic term and "effective potential" $V_{eff}(g_R)$. $\therefore$ g_R is a scalar field, the scalaron.

For consistency w/ the early Universe (CMB, BBN, etc.), the theory should reduce to $\simeq$ GR at early times $\Rightarrow$

$$|g \ll R| \quad \text{and} \quad |g_R| \ll 1 \quad \text{at large } R$$

In this limit extremum of V_{eff} lies at the GR value $R = 8\pi G(\rho_m - 3p_m)$. This is a min or max of V_{eff} depending on the sign of

$$m^2_{g_R} \equiv \frac{\partial^2 V_{eff}}{\partial g_R^2} = \frac{1}{3}\left[\frac{1+g_R}{g_{RR}} - R\right]$$

At large R, when $|R g_{RR}| \ll 1$ and $g_R \to 0$, we have

$$m^2_{g_R} \simeq \frac{1+g_R}{3g_{RR}} \simeq \frac{1}{3g_{RR}}$$

For the scalaron not to be tachyonic, $m^2_{g_R} > 0$, must require

$$\underline{g_{RR} > 0 \text{ at high } R}$$

This is required to keep the early-Universe evolution stable against small perturbations. Go to p. 91 and return

The scalaron mediates an attractive "fifth force" with a range determined by its Compton wavelength:

$$\lambda_c = \frac{2\pi}{m_{g_R}}.$$

λ_c is large at current epochs. $\Rightarrow$ solar system tests and local tests of the Equivalence Principle. In some cases, however, g_R can acquire a large mass in regions of high matter density (galaxies, solar system, ...) $\Rightarrow$ reduced λ_c $\Rightarrow$ force is "screened". This is called the <u>Chameleon mechanism</u>. (return to this).

It's easier to see the extra-force phenomenon in the Einstein frame. We showed how to transform to this frame last time. If we now include other (matter) fields, then we have:

$$S = \int d^4x \sqrt{-g} \left[f(R) + \mathcal{L}_m (\chi, g_{\mu\nu}) \right]$$

matter fields

in the original frame, transforming to (in the Einstein frame)

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g_E}\, R_E + \int d^4x \sqrt{-g_E} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\}$$

(Note: g_E = determinant of metric in Einstein frame, nothing to do w/ g_R)

$$+ \int d^4x \sqrt{-g_E}\, \underline{\underline{e^{-2a\varphi}}}\, \mathcal{L}_m (\chi, e^{-b\varphi} g^E_{\mu\nu})$$

where $a = \sqrt{\frac{16\pi G}{3}}$ and $V(\varphi) = \frac{(R g_R - g)}{16\pi G (1+g_R)^2}$

= Theory of canonical scalar field + GR

In the Einstein frame, the Einstein eqns are

$$G_{\mu\nu} = 8\pi G \left[T^{E,m}_{\mu\nu} + T^{\varphi}_{\mu\nu} \right]$$

and acceleration is "caused" by the potential energy $V(\varphi)$. In the Einstein frame, there is now a coupling between matter (χ) and φ : if the matter behaves as a perfect fluid, then

$$\ddot{\varphi} + 3H_E \dot{\varphi} + \frac{\partial V}{\partial \varphi} = \frac{a}{2} (\rho^E_m - 3p^E_m)$$

long-range interaction

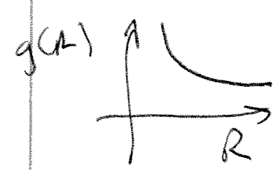
$$\dot{\rho}^E_m + 3H_E (\rho^E_m + p^E_m) = -\frac{a}{2} \dot{\varphi} (\rho^E_m - 3p^E_m)$$

$\therefore$ Particles don't follow geodesics of $g^E_{\mu\nu}$, and there is exchange of energy between φ and matter unless the latter is pure radiation ($p_r = \frac{1}{3}\rho_r$).

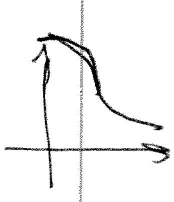
Additional constraints:

- $1+g_R > 0$ $\forall R$: since (loosely) $G_{eff} = \frac{G}{1+g_R}$ and don't want it to change sign: $(1+g_R)R_{\mu\nu} - \frac{1}{2}(R+g)g_{\mu\nu} - \nabla_\mu\nabla_\nu(1+g_R) + \Box(1+g_R)g_{\mu\nu} = 8\pi G T_{\mu\nu}$

- $g_R < 0$: BBN + CMB $\Rightarrow$ $\frac{g(R)}{R}$ and $g_R \to 0$ as $R \to \infty$ (g → constant). Together w/ $g_{RR} > 0$ this implies $g_R < 0$: g_R is a monotonically increasing fn. of R that asymptotes to 0 from below.



- g_R small at recent epochs, to satisfy galactic + solar system tests; (i.e. to allow the chameleon mechanism:) g → const as R → 0
$$|g_R| \leq 10^{-5} \text{ at small } R$$

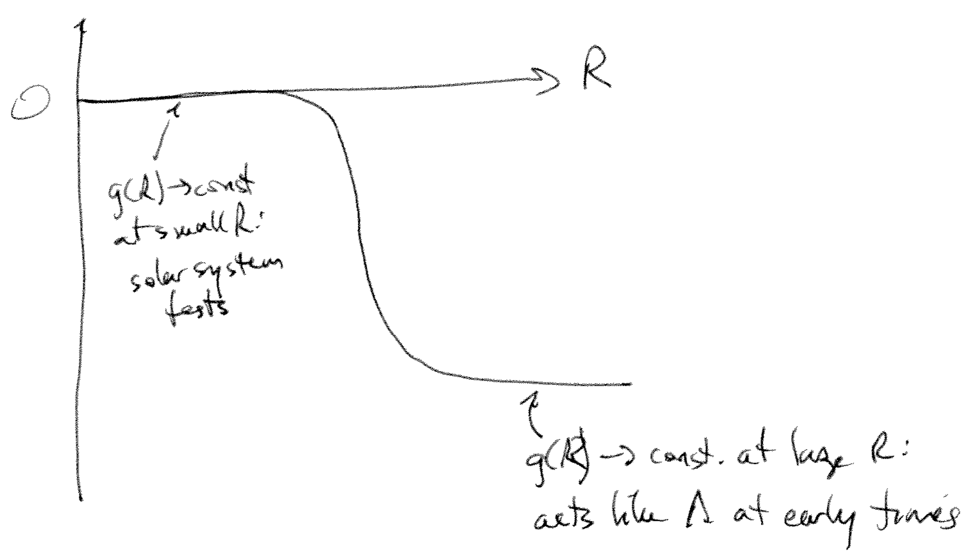


- Models may still have difficulties for strong grav. fields, e.g. existence of neutron stars.

- Message: difficult to mess w/ GR in a simple way.

Example:

Cf. Hu + Sawicki 0705.1158



Solar System Tests:

General
Static, spherically symmetric metric has the form

$$ds^2 = \left(1 - 2A(r) + 2B(r)\right) dt^2 - \left[1 + 2A(r)\right]\left(dr^2 + r^2 d\Omega^2\right)$$

in the weak-field limit $\left(|A|, |B| << 1\right)$. In GR, $B=0$.
Deviations from GR are characterized by

$$\gamma - 1 \equiv \frac{B}{A-B}.$$

Solar system tests: Cassini mission: $|\gamma - 1| < 2.3\times10^{-5}$

- In $f(R)$ models, in the limit $|g_R| << 1$, $\left|\frac{g}{R}\right| << 1$, we have, for $p=0$,

$$\nabla^2 B \simeq \frac{1}{3}\left(8\pi G\rho - R\right)$$ ← source is deviation of R from its GR value

$$\simeq -\nabla^2 g_R$$

$$\Rightarrow B(r) = -\left[g_R(r) - g_{R\infty}\right] \quad \text{assuming } B\to 0 \text{ as } r\to\infty$$

Two classes of solutions:

- high-curvature: $R \simeq 8\pi G\rho$ like GR, and $\nabla^2 g_R << 8\pi G\rho$ has low gradient
- low curvature: $R << 8\pi G\rho$, and $\nabla^2 g_R \simeq -\frac{8\pi G\rho}{3}$ high gradient

For the high-curvature soln, we have $B \simeq 0$. This can happen if m_R becomes large → field gradients are suppressing by $e^{-m_R r}/r$. in presence of ρ ⟷ <u>chameleon mechanism</u>

Braneworld Gravity : DGP Model

- In string theory, one invokes extra spatial dimensions in order to have a consistent theory. In the old days, it was assumed that the extra spatial dimensions had to be very small to avoid experimental bounds; e.g. an N-dimensional sphere of very small radius $R \ll$ physical scales probed by expt.

- Recently, much interest in the idea that the extra dims could be large (macroscopic) but that they're unobserved because all the particles of the Standard Model are dynamically confined to a 3-dim membrane in this higher dimensional space (or a 3+1-dim. membrane in this higher dimensional spacetime).

- Dvali, Gabadadze, and Porrati formulated a model of a 4D brane in 5D Minkowski spacetime, in which the graviton can "leak off" the brane at large scales ⟺ weakens gravity ⟹ can give rise to acceleration.

- The action for this setup is

intrinsic induced curvature on the brane

$$S_5 = \frac{M^3}{16\pi}\int d^5x\,\sqrt{-g}\,R + \frac{M_{Pl}^2}{16\pi}\int d^4x\,\sqrt{-g^{(4)}}\left[R^{(4)} - \frac{16\pi}{M_{Pl}^2}\mathcal{L}_{matter}\right]$$

5-d metric

5-d Planck mass

induced metric on the brane

particles live on the brane: no Λ on the brane

(Brane curvature acts as source for 5D gravity)

Weak-field limit:

$$g_{AB} = \eta_{AB} + h_{AB} \qquad \text{with } |h_{AB}| \ll 1$$

η_{AB}: 5-d Minkowski metric

h_{AB}: small perturbation

$A, B = 0, 1, 2, 3, 4$

Fourier decompose: $h_{AB}(\vec{x}, t) = \int d^3p \; e^{-i\vec{p}\cdot\vec{x}} \, h_{AB}(\vec{p}, t)$

$T_{AB}(\vec{x}, t) = \quad " \quad " \quad T_{AB}(\vec{p}, t)$

Consider a static energy-momentum source confined to the brane: $T_{\mu\nu}(p)$. Can solve weak-field equs. for S_5:

this would be $\frac{1}{2}$ in GR

$$h_{\mu\nu}(p) = \frac{8\pi G}{p^2 + 2(G/G^{(5)})p} \left(T_{\mu\nu}(p) - \frac{1}{3}\eta_{\mu\nu} T^{\alpha}_{\alpha}(p) \right)$$

where $\mu, \nu = 0, 1, 2, 3$ live on the brane, $G = \frac{1}{m_{Pl}^2}$, and $G^{(5)} = 1/M^3$. Characteristic "cross-over" distance given by

$$r_c = \frac{1}{2}\left(\frac{G^{(5)}}{G}\right) = \frac{m_{Pl}^2}{2M^3}$$

That is, for modes w/ $p \gg r_c^{-1}$, $h_{\mu\nu}(p) \sim p^{-2}$, which can show implies the usual grav. potential $V(r) \sim \frac{1}{r}$ for $r \ll r_c$ (inverse-square law). But for $p \ll r_c^{-1}$, have $h_{\mu\nu} \sim p^{-1}$, which implies $V(r) \propto r^{-2}$ for $r \gg r_c$: $F_{grav} \sim \vec{\nabla} V \sim 1/r^3$, gravity is weaker at $r > r_c$ because it leaks off the brane into 5D.

- Physically, the presence of stress-energy ($T_{\mu\nu}$) on the brane provides an energy cost for gravitons w/ wavelengths $\lambda < r_c$ to propagate into the 5D "bulk", but longer-wavelength gravitons are free to propagate into the bulk.

- In fact, deviations from GR persist down to scales smaller than r_c, down to the effective radius $r_* = (r_g r_c^2)^{1/3}$, where

$$r_g = \frac{2GM}{c^2} = \text{Schwarzschild radius}.$$

This is because there are $A, B = 4$ components (5D) (bulk) of h_{AB} that are large that we've ignored. $\Rightarrow$ There are 3 regimes:

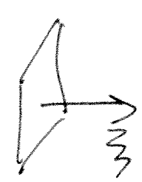
1.) $r \ll r_*$: close to GR
2.) $r_* \ll r \ll r_c$: scalar-tensor-like theory
3.) $r \gg r_c$: 5D gravity $V \sim 1/r^2$

Lecture 13:

- Cosmological solutions:
  - Assume $\mathcal{L}_m$ describes an ideal fluid on the brane w/ density ρ and pressure $p \geq 0$ (no dark energy).
  - Cosmological Metric takes the form

$$ds^2 = N^2(t,\xi)\,dt^2 - A^2(t,\xi)\,dx^i dx^j - B^2(t,\xi)\,d\xi^2$$

3-space dimensions (assuming $k=0$) — 5th dimension



- Solve Einstein eqns. for N, A, B. The 4d scale factor, $a(t) = A(t, \xi = 0)$ (i.e. the brane is at $\xi = 0$) satisfies

$$H^2 \pm \frac{H}{r_c} = \frac{8\pi G}{3}\rho$$

Can generalize this to include 3-space curvature k.