- <u>Convergence</u>: isotropic magnification of angular size $\bigcirc \to \bigcirc$

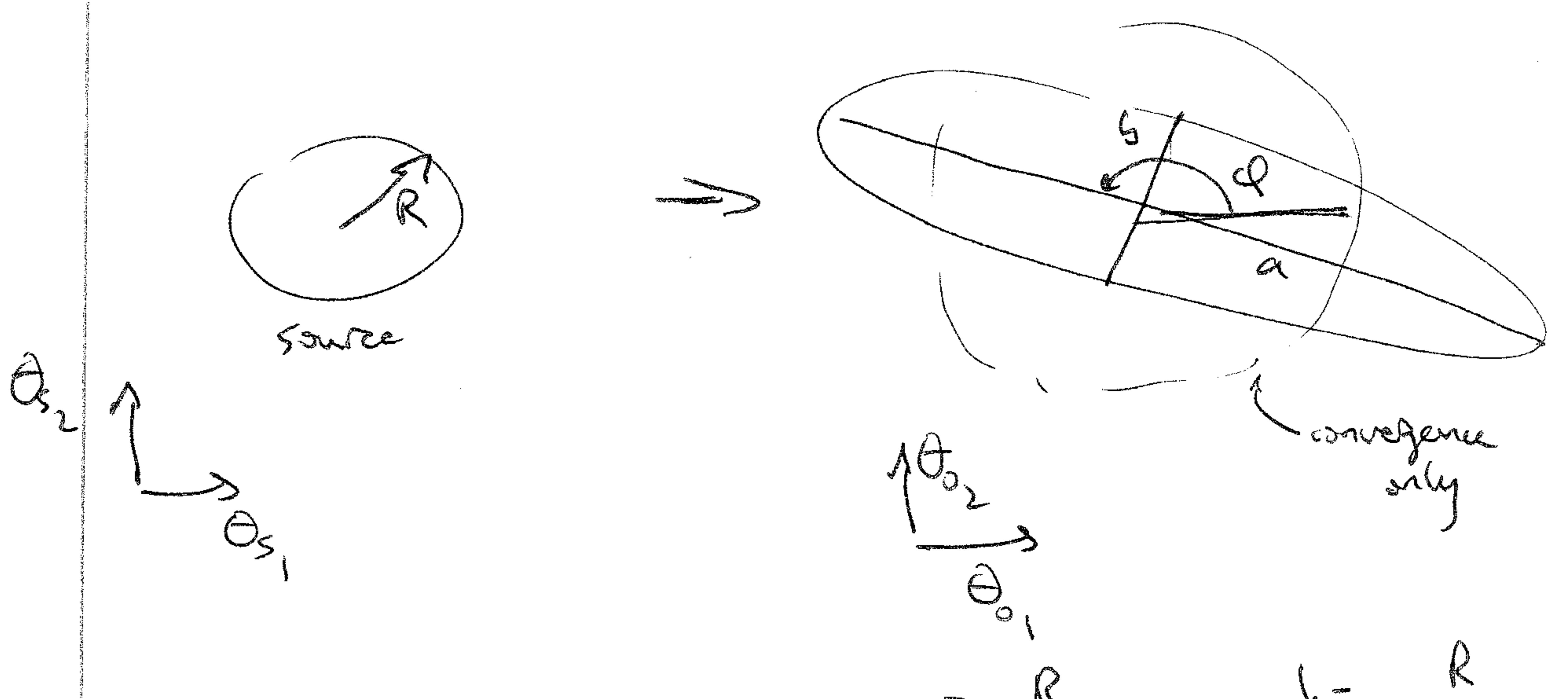
- <u>shear</u>: anisotropy of the mapping: $\bigcirc \to 0$

For small circular source, the lensed image is an ellipse, with major and minor axes

$$a = (1-\kappa-\gamma)^{-1}, \quad b = (1-\kappa+\gamma)^{-1}$$

∴ Shear can be estimated from galaxy shapes.

- Convergence can be estimated from galaxy sizes or numbers.



$$a = \frac{R}{1-\kappa-\gamma} \qquad b = \frac{R}{1-\kappa+\gamma}$$

$$\gamma = \gamma_1 + i\gamma_2 = |\gamma| e^{i2i\varphi}$$

$$\frac{b}{a} = \frac{1-|g|}{1+|g|} \quad \text{where } g = \frac{\gamma}{|1-\kappa|}$$ reduced complex shear

On p. 83a, we saw that the amplification μ satisfies

$\mu = \left|\frac{\theta\, d\theta}{\beta\, d\beta}\right|$, for an axisymmetric lens. Comparing with above, we find that generally the amplification

$$\mu = \frac{1}{\det A} = \left|\frac{\beta\, d\beta}{\theta\, d\theta}\right|^{-1} \quad \text{for axisymmetric case}$$

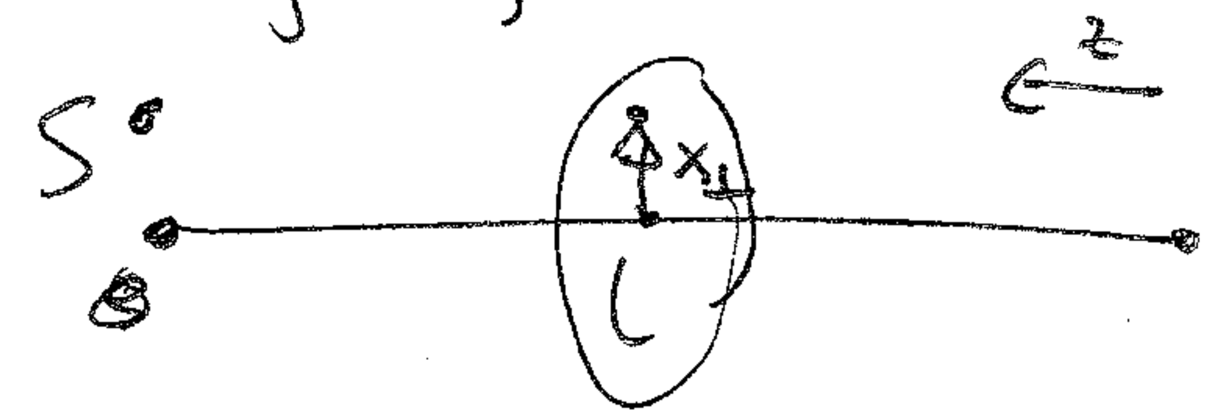
$(\beta = \partial_s)$

$$= \left[(1-\kappa)^2 - |\gamma|^2\right]^{-1} \simeq 1 + 2\kappa \quad \text{for } \kappa, \gamma \ll 1$$

where $\gamma = \gamma_1 + i\gamma_2$

Let's now consider general properties of lensing by

Axisymmetric Mass Distributions:



In this case, $\Sigma(\vec{x}_\perp) = \Sigma(x_\perp)$, where $x_\perp = |\vec{x}_\perp|$ = dist. of a point from lens center

$$= \int \rho(r)\, dz$$

For thin axisymmetric lens, the bend angle

$$\alpha(x_\perp) = \frac{4GM(x_\perp)}{c^2 x_\perp}$$

where (projected) mass within tube of radius $x_\perp$ is

$$M(x_\perp) = 2\pi \int dx'_\perp\, x'_\perp\, \Sigma(x'_\perp)$$

This is analog of Newton's theorem: for axisymmetric system, bend angle only depends on enclosed mass and is equivalent to putting all that mass at the central point. (no external shear)

Axisymmetric Lens Equation: everything reduces to 1-d:

impact parameter $b = \theta D_L$

$$\theta - \beta = \frac{D_{LS}}{D_S}\alpha(\theta) = \frac{D_{LS}}{D_S}\frac{4GM(b)}{c^2}\left(\frac{1}{D_L\theta}\right)$$

$\theta = \theta_I$
$(\beta = \theta_S)$

$$\therefore \quad \beta(\theta) = \theta\left[1 - \frac{4GM(b)}{c^2\theta^2}\frac{D_{LS}}{D_S D_L}\right]$$

Now use $\theta = b/D_L$ and $\Sigma_{crit} = \frac{D_S c^2}{4\pi G D_L D_{LS}}$

$$= \theta\left[1 - \frac{M(b)}{\pi b^2 \Sigma_{crit}}\right]$$

$$= \theta\left[1 - \frac{\bar{\Sigma}(<b)}{\Sigma_{crit}}\right] \equiv \theta\left[1 - \bar{K}(\theta)\right]$$

where $\bar{\Sigma}(<b)$ = average surface mass density inside radius b.

- Recall that, for $\beta = 0$, the image is a ring at the Einstein radius. From above equ., at $\beta = 0 \Leftrightarrow b = b_E = D_L\theta_E$, have

$$\bar{\Sigma}(<b_E) = \Sigma_{crit} \quad \Rightarrow \quad \Sigma_{crit} = \frac{M(b_E)}{\pi b_E^2}$$

$\therefore \Sigma_{crit}$ is mean surface density within b_E.

Amplification for Axisymmetric Lens:

Return to the diagram on p. 83a. We saw there that a circular source is radially stretched by an amount

$$\frac{\Delta x}{\Delta y} = \frac{d\theta}{d\beta} \quad \text{and tangentially stretched by} \quad \frac{x\Delta\phi}{y\Delta\phi} = \frac{\theta}{\beta}$$

Tangential Shear

measures the difference between the compression (inverse stretching) in the radial and tangential directions:

$$\gamma_T \equiv \frac{1}{2}\left(\frac{d\beta}{d\theta} - \frac{\beta}{\theta}\right)$$

Use the lens equation above for axisymmetric systems to evaluate this:

$$\gamma_T = \frac{1}{2}\left[1 - \frac{M(b)}{\pi b^2 \Sigma_{crit}}\right] - \frac{\theta}{2}\frac{d}{d\theta}\Bigg[\underbrace{\frac{M(b)}{\pi b^2 \Sigma_{crit}}}_{b^2 = \theta^2 D_L^2}\Bigg] - \frac{1}{2}\left[1 - \frac{M(b)}{\pi b^2 \Sigma_{crit}}\right]$$

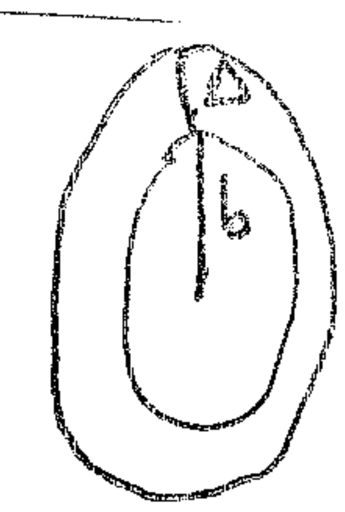
cancel

$$= \frac{1}{\pi \Sigma_{crit}}\frac{M(b)}{b^2} - \frac{1}{2\pi \Sigma_{crit}\,\theta D_L^2}\underbrace{\frac{dM(\theta D_L)}{d\theta}}_{= D_L \frac{dM(b)}{db}}$$

$$= \frac{M(b)}{\pi b^2 \Sigma_{crit}} - \frac{1}{2\pi b \Sigma_{crit}}\frac{dM}{db}$$

Now also use

$$\Sigma(b) = \frac{M(b+\Delta) - M(b)}{\pi\left[(b+\Delta)^2 - b^2\right]}$$ for thin annulus of thickness Δ



$$\simeq \frac{\frac{dM}{db}\Delta}{\pi\, 2\Delta b}$$ using $(b+\Delta)^2 - b^2 = b^2\left[\left(1+\frac{\Delta}{b}\right)^2 - 1\right] \simeq b^2\left(1 + \frac{2\Delta}{b} - 1\right)$

$$= \frac{dM/db}{2\pi b}$$

$\therefore$ we have $$\boxed{\gamma_T = \frac{\bar{\Sigma}(<b) - \Sigma(b)}{\Sigma_{crit}}}$$ ← we'll return to this in weak lensing

Cosmic Shear + Galaxy–Galaxy lensing:

- Cosmic shear: correlation of shear field $\gamma(\vec{x})$ with itself
- Galaxy-galaxy: " " " " " " foreground galaxy distribution.

Convergence field:

$$K(\vec{\theta}_0) = \frac{1}{c^2}\int_0^{\chi_h} d\chi\, \tilde{g}(\chi)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\Phi(\vec{x}_\perp,\chi)$$

$\delta = (\rho(\vec{x})-\bar{\rho})/\bar{\rho}$

Using Poisson equ. $\longrightarrow$ $\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)\Phi = 4\pi G\bar{\rho}_m\delta$ ~~[illegible]~~

$= 4\pi G\delta\rho_c\Omega_m = 4\pi G\delta\Omega_m\frac{3H^2}{8\pi G} = \frac{3}{2}H^2\Omega_m\delta$

$$= \frac{3}{2}\left(\frac{H_0}{c}\right)^2\Omega_{m,0}\int_0^{\chi_h}\underbrace{\frac{\tilde{g}(\chi)}{a(\chi)}}_{\text{radial window function}}\delta\left[S_k(\chi)\vec{\theta}_0,\chi\right]d\chi$$

where $\tilde{g}(\chi) = S_k(\chi)\int_\chi^{\chi_h}\frac{S_k(\chi'-\chi)}{S_k(\chi')}\underbrace{P(\chi')}_{\text{source redshift distn.}}d\chi'$

Then 2pt. fn. of K $\longrightarrow$ $\int$ two-pt fn. of δ ~ $P_k(\delta)$

see slides

$$\Delta_K^2(k) = \Delta_\gamma^2(k) = k^3P_\gamma(k) = \frac{9\pi}{4}\left(\frac{H_0}{c}\right)^4\Omega_{m,0}^2\int_0^{\chi_h}\frac{g^2(\chi)}{a^2(\chi)}k^3P_\delta(k;\chi)\frac{d\chi}{k}$$

$\longleftrightarrow$ Angular power spectrum:

$$C_\ell^{\gamma\gamma} = \int dz\,\frac{H(z)}{D_A^2(z)}W^2(z)P_\delta\left(k=\frac{\ell}{D_A};z\right) \qquad W(z) = \frac{3}{2}\Omega_m\frac{H_0}{H}\frac{H_0D_{0c}}{a}\int_z^\infty dz'\frac{D_{LS}}{D_{0S}}W_g(z')$$

Measuring Shear

- Measure shape of galaxies through 2nd moments of flux:

$$Q_{ij} = \int x_i x_j \, \omega(x) \, I(x_1, x_2) \, dx_1 dx_2$$

surface brightness

→ ellipticities:

$$\varepsilon_1 = \frac{Q_{11} - Q_{22}}{Q_{11} + Q_{22}} \qquad \varepsilon_2 = \frac{2 Q_{12}}{Q_{11} + Q_{22}}$$

compression along x_1, x_2 compression along ↙↗ ↖↘

For circular source $\varepsilon_1 =$ estimate of γ_1 w/ noise.
$(Q_{11} = Q_{22},\ Q_{12} = 0)$

but galaxies aren't circular: beat down noise by estimating shear in a patch w/ noise $\frac{\langle \varepsilon^2 \rangle}{\sqrt{N}}$

DES: $n \sim 10$ gals / arcmin^2. for $r \sim 24$

Signal is $\sim 1\%$ → determines spatial / angular resolution

→ multiple areas to beat down noise

Next: numbers, estimates, systematics → stars: PSF interpolation + correction

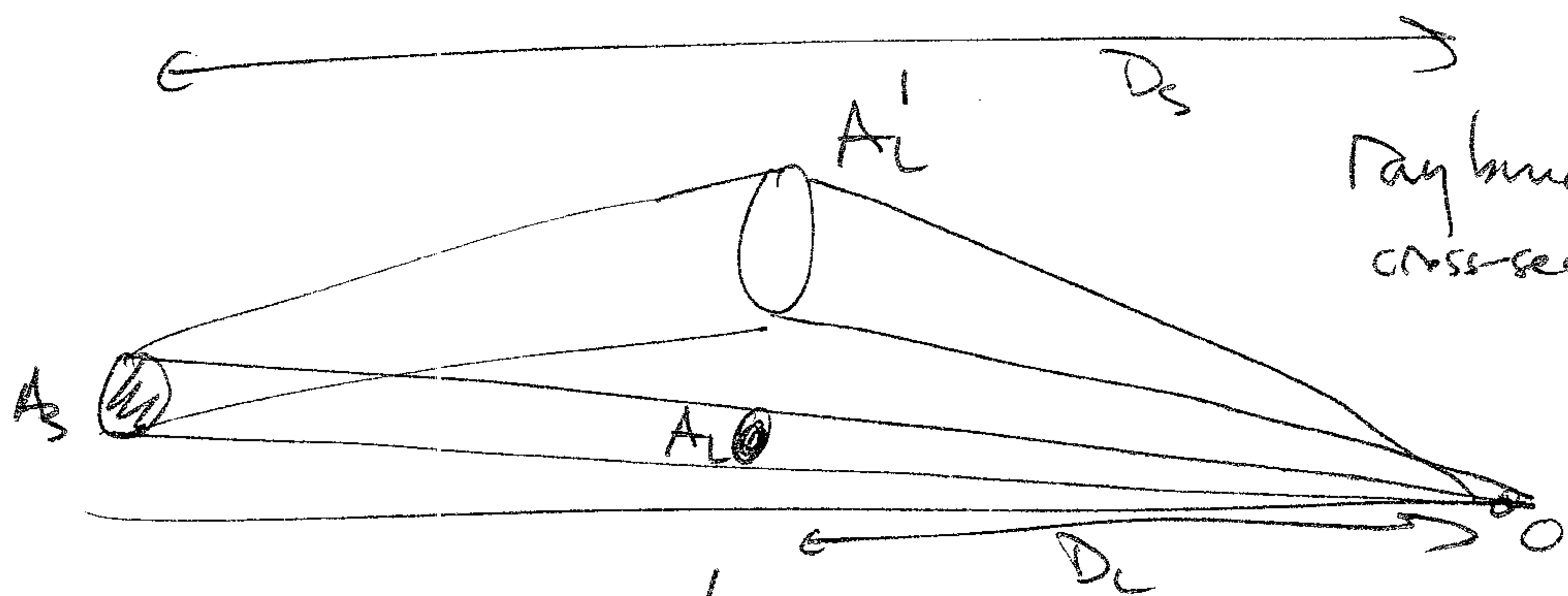
Shear + Ellipticities:

- Cons. of surface Brightness: Liouville's Thm. + absence of photon emission or absorption.

  (surface brightness = # photons / time / area)

$\therefore$ Image area > Source area + SB cons $\rightarrow$ larger photon flux

This arises from grav. focusing of ray bundles:



Solid angle: $\frac{A_L'}{D_L^2} > \frac{A_L}{D_L^2} = \frac{A_S}{D_S^2}$

- Focusing $\rightarrow$ more photons reach O.
- But SB just dictated by rate of γ emission from source

Larger solid angle + conserved SB $\rightarrow$ amplification.

Think of Einstein ring: many more γ's reach O due to focusing

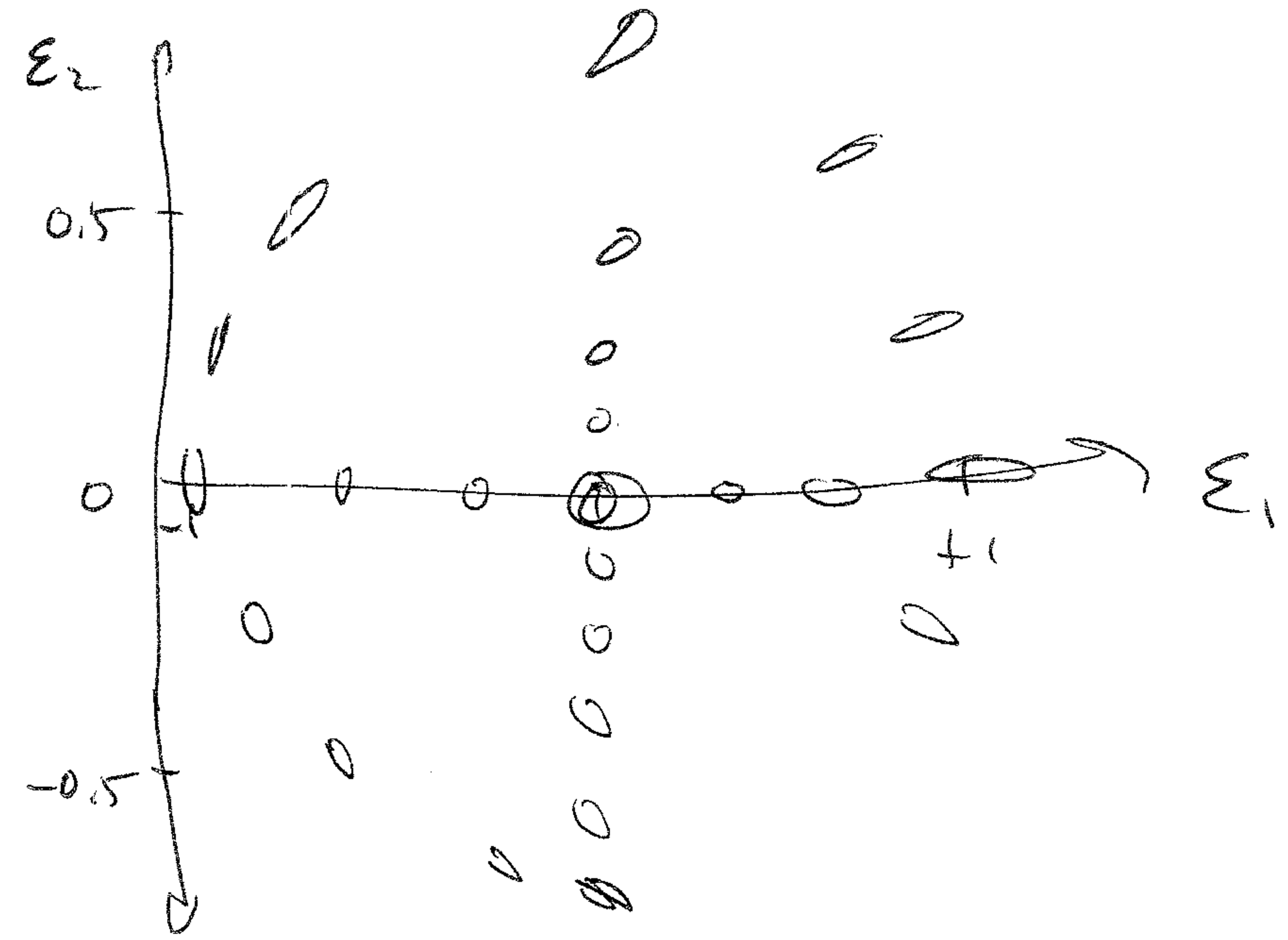
SB Cons: SB $I_L(\vec{\theta}_0) = I_S(\vec{\theta}_S(\vec{\theta}_0))$

(I_L: lens plane; I_S: source plane)

$$\overleftrightarrow{A}(\vec{\theta}_0) = \frac{\partial \vec{\theta}_S}{\partial \vec{\theta}_0} \qquad (2\times 2 \text{ matrix}) \qquad \vec{\theta}_0^{(*)} \leftrightarrow \vec{\theta}_S^{(*)}(\vec{\theta}_0^{(*)})$$

$$I_L(\vec{\theta}_0) = I_S\left[\vec{\theta}_S^{(*)} + \overleftrightarrow{A}(\vec{\theta}_0)\cdot(\vec{\theta}_0 - \vec{\theta}_0^{(*)})\right] \quad \text{to linear order}$$

Shape of image ellipses of circular source.



Intrinsic orientation of (non-round) galaxies assumed

random: $\langle \varepsilon^{(source)} \rangle = \langle \varepsilon_1^{(s)} + i\varepsilon_2^{(s)} \rangle$

Then $\langle \varepsilon^{image} \rangle = g = \dfrac{\gamma}{|1-\kappa|}$ for $|g| \leq 1$ (WL)

Systematics:

- PSF anisotropy: variety of effects cause distortions (cf. Kent)
- Shear calibration error: seeing makes things look round, especially small things. Need to calibrate shear → ellipticity as fn of source size.

See Buckley-Geer et al.