Lecture 6:

Weak Gravitational Lensing

~~[illegible]~~

- FRW metric:

$$ds^2_{FRW} = c^2 dt^2 - a^2(t)\left(d\chi^2 + S_k^2(\chi)\underbrace{\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}_{=d\omega^2}\right)$$

$$S_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & 0 \\ \sin\chi & +1 \end{cases}$$

$$ds^2_{FRW} = a^2(\tau)\left[d\tau^2 - \left(d\chi^2 + S_k^2(\chi)\, d\omega^2\right)\right]$$

where $\tau = \int \frac{c\,dt}{a(t)}$ is conformal time.

Include spatial perturbations in density:

$$ds^2 = a^2(\tau)\left[\left(1 + \frac{2\Phi(\vec{x},\tau)}{c^2}\right)d\tau^2 - \left(1 - \frac{2\Phi}{c^2}\right)dl^2\right]$$

where $dl^2 = d\chi^2 + S_k^2(\chi)\, d\omega^2$

Φ satisfies Poisson eqn: $\nabla^2\Phi = 4\pi G\rho(\vec{x},\tau)$

$\Rightarrow$ light propagates as if in medium w/ spatially varying index of refraction:

$$n = 1 - \frac{2\Phi(\vec{x},\tau)}{c^2}$$

Density enhancement $\Rightarrow \Phi \leq 0 \Rightarrow n \geq 1$.

$$\frac{dl}{d\tau} = c\left(1 + \frac{2\Phi}{c^2}\right)$$

$\Rightarrow$ light travel time $\tau = \frac{1}{c}\int dl\, n = \frac{1}{c}\int dl\left(1 - \frac{2\Phi}{c^2}\right)$

~~Gradient~~
- Fermat's theorem: photon trajectories are paths that extremize light travel time

- Gradients in $\Phi \Longleftrightarrow$ gradients in $n \Rightarrow$ bending of light rays.

- Geodesic equ. of motion:

$$\frac{d^2\vec{x}}{d\tau^2} = -\frac{2}{c^2}\vec{\nabla}\Phi$$

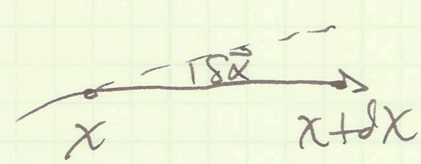
(famous factor of 2 relative to Newtonian gravity).

- For photon with spatial direction $\hat{u}$, to 1st order in Φ this implies

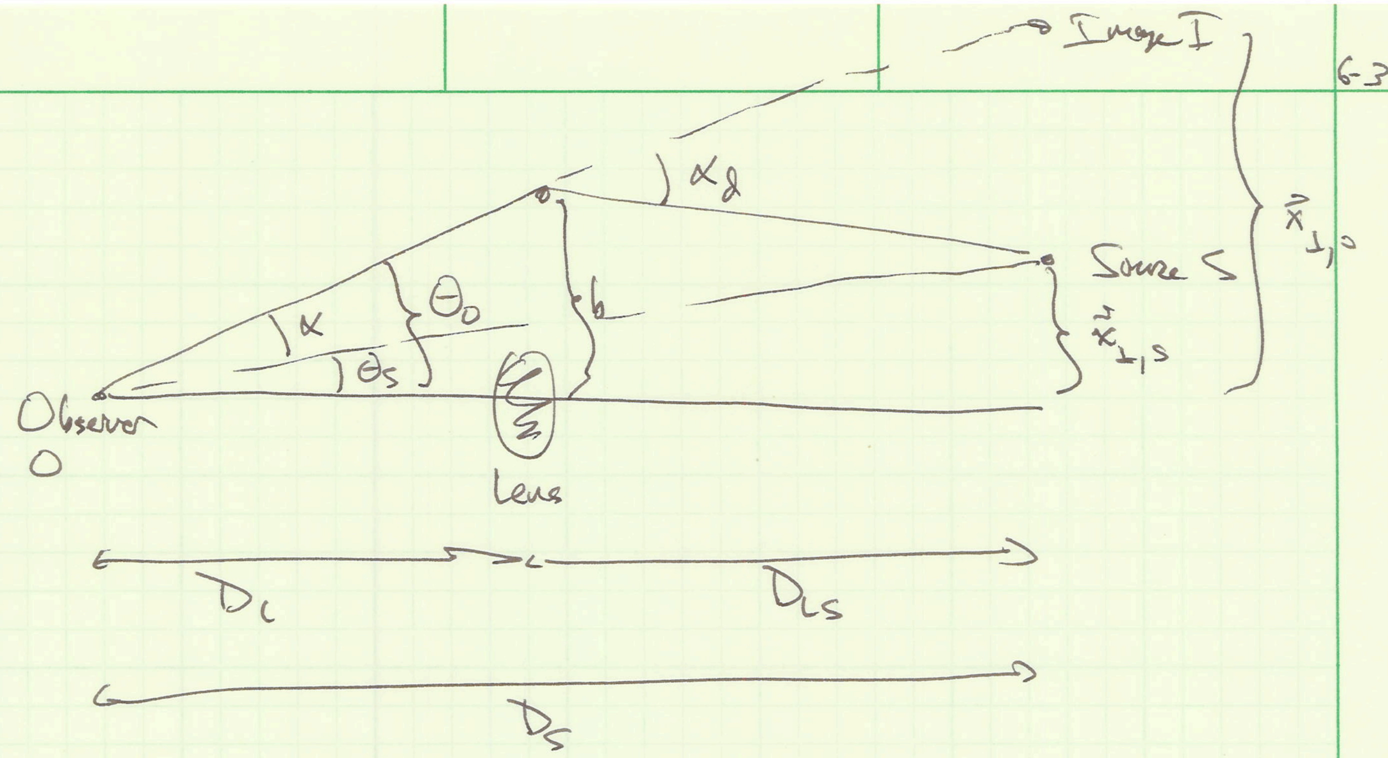
$$\frac{d\hat{u}}{d\chi} = -\frac{2}{c^2}\vec{\nabla}_\perp \Phi$$ where derivative is taken in directions $\perp$ to path

$\therefore$ Angular deflection as it propagates from χ to $\chi + d\chi$ is

$$\delta\vec{\alpha}_d = -\frac{2}{c^2}\vec{\nabla}_\perp \Phi\, d\chi$$



- For small bend angles, evaluate this along the unperturbed path.



Deflection leads to change in image position at χ_s by

$$\delta\vec{x}_\perp(\chi) = S_k(\chi_s-\chi)\,\delta\vec{\alpha}_d(\chi)$$

comoving angular-diameter distance from χ_s to χ

$\Rightarrow$ Image position seen by O is

$$\vec{x}_{\perp,0} = \vec{x}_{\perp,S} - \frac{2}{c^2}\int_{\chi_s}^{0} S_k(\chi_s-\chi)\,\vec{\nabla}_\perp\Phi(\chi)\,d\chi$$

Use $\vec{x}_{\perp,0} = \vec{\theta}_0 \cdot S_k(\chi_s)$, $\vec{x}_{\perp,S} = \vec{\theta}_S\, S_k(\chi_s)$ $\Rightarrow$

$$\vec{\theta}_S = \vec{\theta}_0 - \frac{2}{c^2}\int_0^{\chi_s} \frac{S_k(\chi_s-\chi)}{S_k(\chi_s)}\,\vec{\nabla}_\perp\Phi(\chi)\,d\chi = \vec{\theta}_0 - \vec{\alpha} = \vec{\theta}_0 - \frac{D_{LS}}{D_S}\vec{\alpha}_d$$

Describes mapping from $\vec{\theta}_S$ to $\vec{\theta}_0$: source to image.

This is the lens Equation

~~Notes:~~

- For point mass lens, $\alpha_d = \dfrac{4GM}{bc^2}$, b = impact parameter

- ~~For thin lens (size ...)~~ More generally - Can define

projected surface mass density:

$$\Sigma(\vec{x}) = \int dl\, \rho(\vec{x})$$

Then $\vec{\alpha} = -\vec{\nabla}\Psi$

where Ψ satisfies 2-d Poisson eqn:

$$\nabla^2 \Psi = 2K = \frac{2\,\Sigma(\vec{x})}{\Sigma_{crit}}$$ K = convergence

where $$\Sigma_{crit} \equiv \frac{c^2 D_s}{4\pi G D_L D_{LS}}$$

If $K \geq 1$ in at least one point of the lens, then multiple imaging is in principle possible.

Weak lensing: $K \ll 1$.

Axisymmetric lens:

$$\Sigma(\vec{x}_\perp) = \Sigma(x_\perp)$$

Point mass example:

$$\alpha_d = \frac{4GM}{bc^2} \quad , \quad b = \theta_0 D_L \quad \Rightarrow$$

Lens equation becomes:

$$\theta_s = \theta_0 - \frac{D_{LS}}{D_L D_S} \frac{4GM}{c^2 \theta_0} \equiv \theta_0 - \frac{\theta_E^2}{\theta_0}$$

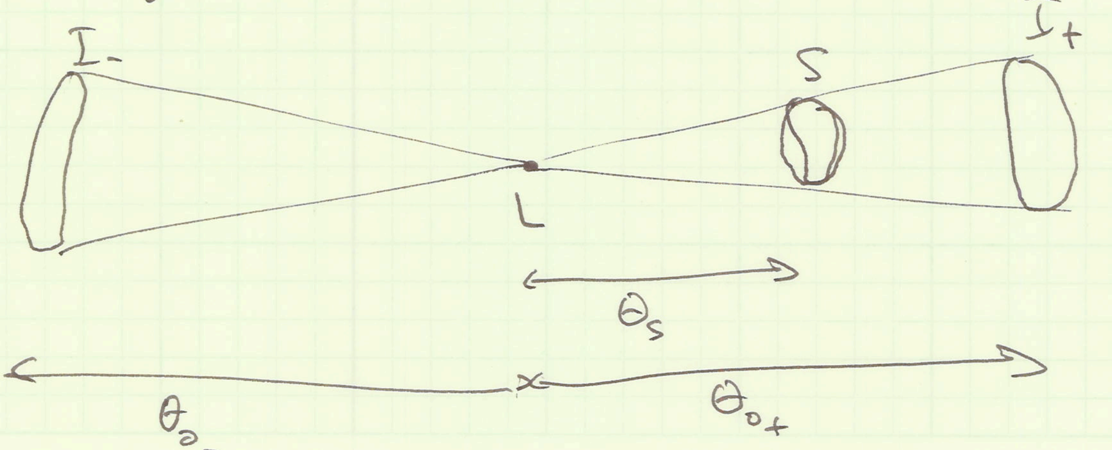
θ_E = Einstein angle.

For given θ_s, this has 2 quadratic solutions:

$$\theta_{0\pm} = \frac{1}{2}\left(\theta_s \pm \sqrt{\theta_s^2 + \theta_E^2}\right)$$

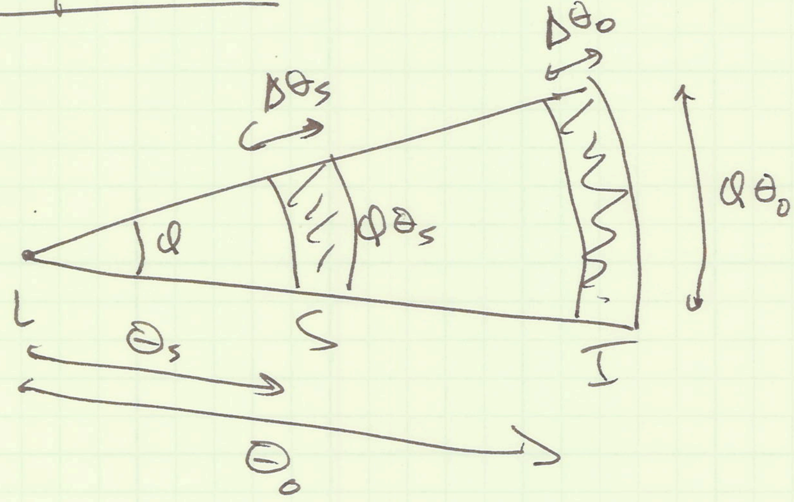
correspond to 2 images on opposite sides of the lens:

Looking along LOS:



Note: for $\theta_s \to 0$, image is a ring of radius θ_E

Amplification:



$\frac{\theta_0}{\theta_s}$ = tangential stretch

$\Delta\theta_0/\Delta\theta_s$ = radial stretch

Ratio of image area to source area is

$$\mu = \frac{A_I}{A_s} = \frac{\phi\,\theta_0\,\Delta\theta_0}{\phi\,\theta_s\,\Delta\theta_s} = \left|\frac{\theta_0\, d\theta_0}{\theta_s\, d\theta_s}\right|$$

This is also the ratio of total flux, since lensing conserves surface brightness. This holds generally for axisymmetric lenses.

Now go back to general lens equation. Jacobian of this mapping is given by:

$i, j = 1, 2$ transverse directions

$$A_{ij}(\vec{\theta}_0, \chi_s) = \frac{\partial \theta_{s_i}}{\partial \theta_{0_j}} = \frac{1}{S_k(\chi_s)} \frac{\partial x_{s_i}}{\partial \theta_{0_j}}$$

$$= \delta_{ij} - \frac{2}{c^2} \sum_k \int_0^{\chi_{horizon}} g(\chi)\, \partial_i \partial_k\, \Phi(\vec{x}_\perp, \chi) A_{kj}(\vec{\theta}_0, \chi)\, d\chi$$

where $g(\chi) = \dfrac{S_k(\chi_s - \chi) S_k(\chi)}{S_k(\chi_s)} H(\chi_s - \chi)$

Heaviside fn.

For $\Phi \ll c^2$, replace A_{kj} in integrand by δ_{kj} $\Rightarrow$

$$A_{ij}(\vec{\theta}_0, \chi_s) = \left(\delta_{ij} - \frac{\partial^2 \Psi}{\partial \theta_{0_i} \partial \theta_{0_j}} \right)$$

where projected potential satisfies

$$\Psi(\vec{x}_\perp, \chi_s) = \frac{2}{c^2} \int_0^{\chi_H} d\chi\, g(\chi)\, \Phi(\vec{x}_\perp, \chi)$$

We can define the components of this symmetric 2x2 matrix as

$$A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

where

$$\kappa(\vec{\theta}_0) = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2} + \frac{\partial^2\psi}{\partial\theta_2^2}\right) \quad \text{convergence}$$

$$\left.\begin{aligned} \gamma_1(\vec{\theta}_0) &= \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2} - \frac{\partial^2\psi}{\partial\theta_2^2}\right) \\ \gamma_2(\vec{\theta}_0) &= \frac{\partial^2\psi}{\partial\theta_1\partial\theta_2} \end{aligned}\right\} \gamma = \gamma_1 + i\gamma_2 \quad \text{traceless shear}$$

<u>Amplification:</u>

$$\mu = \frac{1}{\det A} \quad \left(= \left|\frac{\theta_s\, d\theta_s}{\theta_0\, d\theta_0}\right|^{-1} \right)$$

(for axisymmetric lens)

$$= \left[(1-\kappa)^2 - |\gamma|^2\right]^{-1} \simeq 1+2\kappa \quad \text{for } \kappa, \gamma \ll 1$$

(weak lensing)

Regions in image plane where $\mu \to \infty$: <u>critical curves</u> of ∞ magnification and ∞ stretching

Corresponding regions in source plane = <u>caustics</u>

- Convergence: isotropic magnification of angular size

- Shear: anisotropy of the mapping:

For small circular source, the lensed image is an ellipse, with major and minor axes

$$a = (1-\kappa-\gamma)^{-1}, \quad b = (1-\kappa+\gamma)^{-1}$$

∴ Shear can be estimated from galaxy shapes.

- Convergence can be estimated from galaxy sizes or numbers