Science of the Dark Energy Survey

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Astronomy 41100

Lecture 1, Oct. 1 2010

The Dark Energy Survey

- Study Dark Energy using
  4 complementary* techniques:
  - I. Cluster Counts
  - II. Weak Lensing
  - III. Baryon Acoustic Oscillations
  - IV. Supernovae

- Two multiband surveys:
  - 5000 deg^2 g, r, i, z, Y to 24th mag
  - 15 deg^2 repeat (SNe)

- Build new 3 deg^2 FOV camera
  and Data management sytem
  - Survey 2012-2017 (525 nights)
  - Camera available for community use the rest of the time (70%)

Blanco 4-meter at CTIO

*in systematics & in cosmological parameter degeneracies
*geometric+structure growth: test Dark Energy vs. Gravity

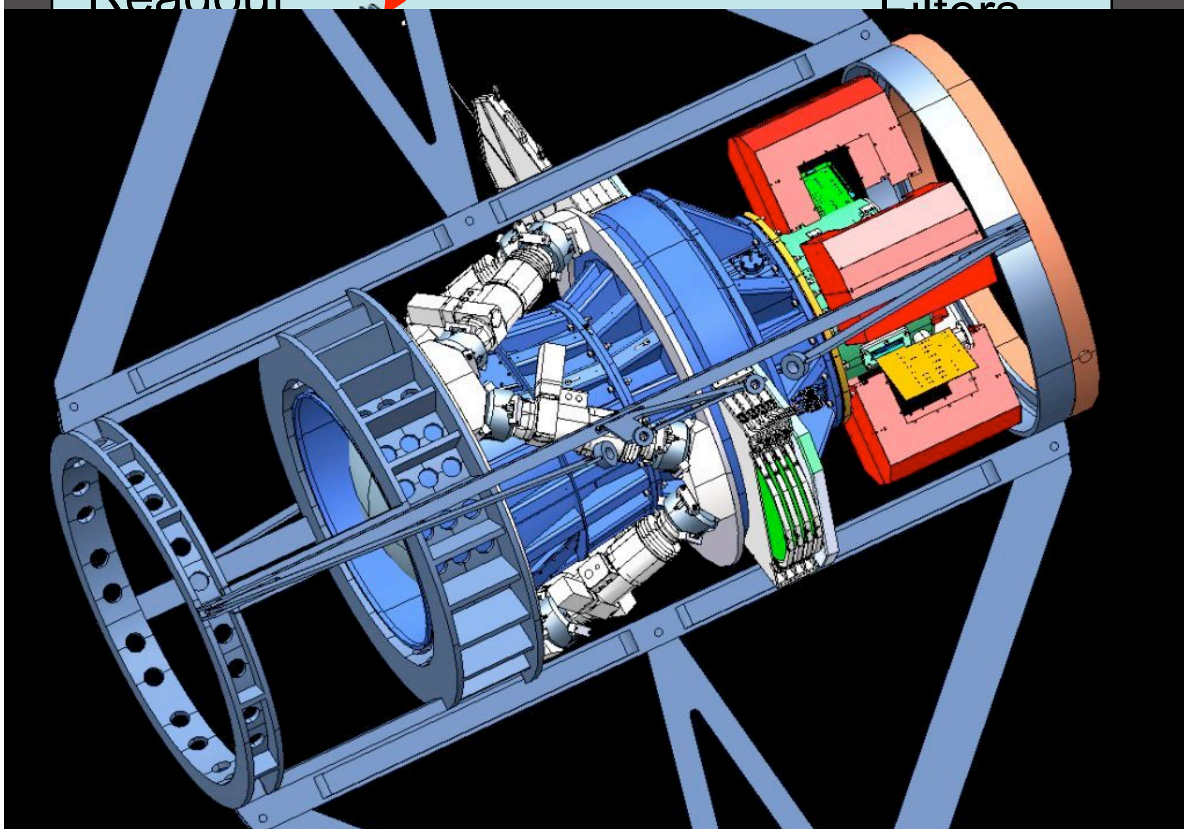
DARK ENERGY SURVEY

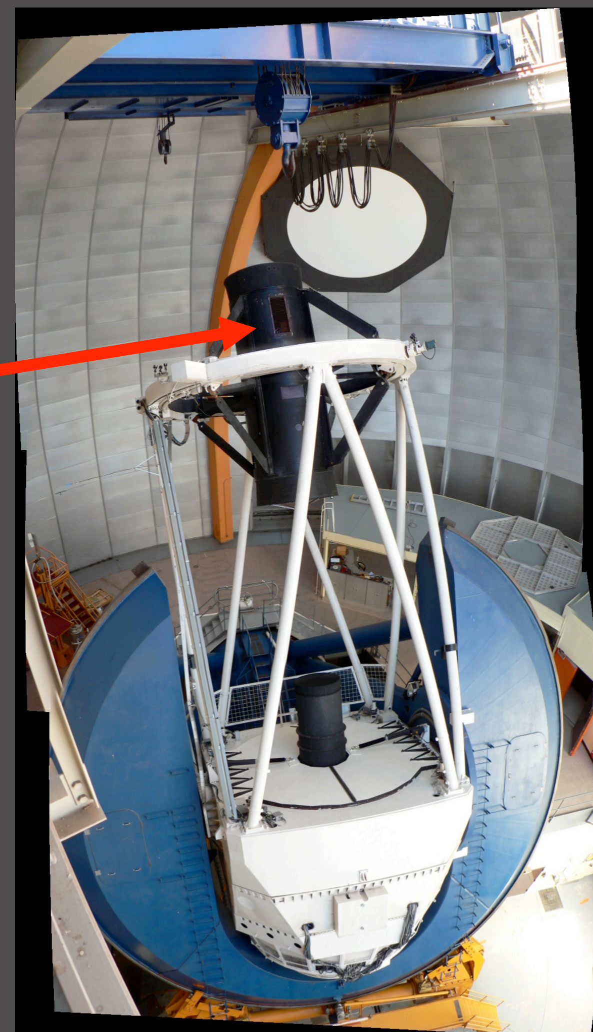
DES Instrument: DECam

CCD Readout

Mechanical Interface of DECam Project to the Blanco

Filters



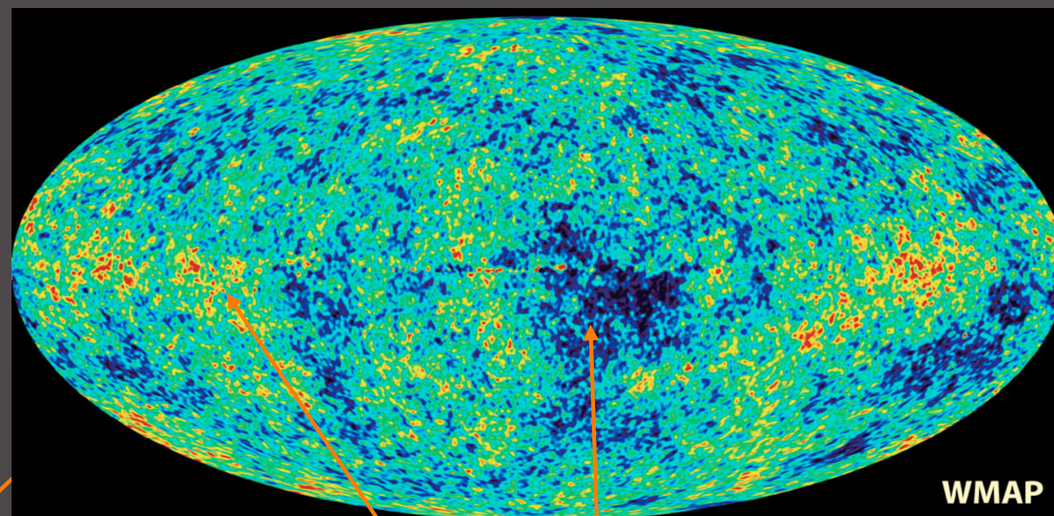


Designed for improved image quality compared to current Blanco mosaic camera

Cosmic Microwave Background Radiation

The Universe is filled with a bath of thermal radiation

COBE map of the CMB temperature



On large scales, the CMB temperature is nearly <u>isotropic</u> around us (the same in all directions): snapshot of the young Universe, t ~ 400,000 years

T = 2.728 degrees above absolute zero

Temperature fluctuations $\delta T/T \sim 10^{-5}$

The Cosmological Principle

- We are not priviledged observers at a special place in the Universe.

- At any instant of time, the Universe should appear

  ISOTROPIC

  (averaged over large scales) to all Fundamental Observers (those who define local standard of rest).

- A Universe that appears isotropic to all FO's is

  HOMOGENEOUS

  the same at every location (averaged over large scales).

The only mode that preserves homogeneity and isotropy is overall expansion or contraction:

Cosmic scale factor $a(t)$

Model completely specified by *a(t)* and sign of spatial curvature

On average, galaxies are <u>at rest</u> in these expanding (comoving) coordinates, and they are not expanding--they are gravitationally bound.

Wavelength of radiation scales with scale factor:

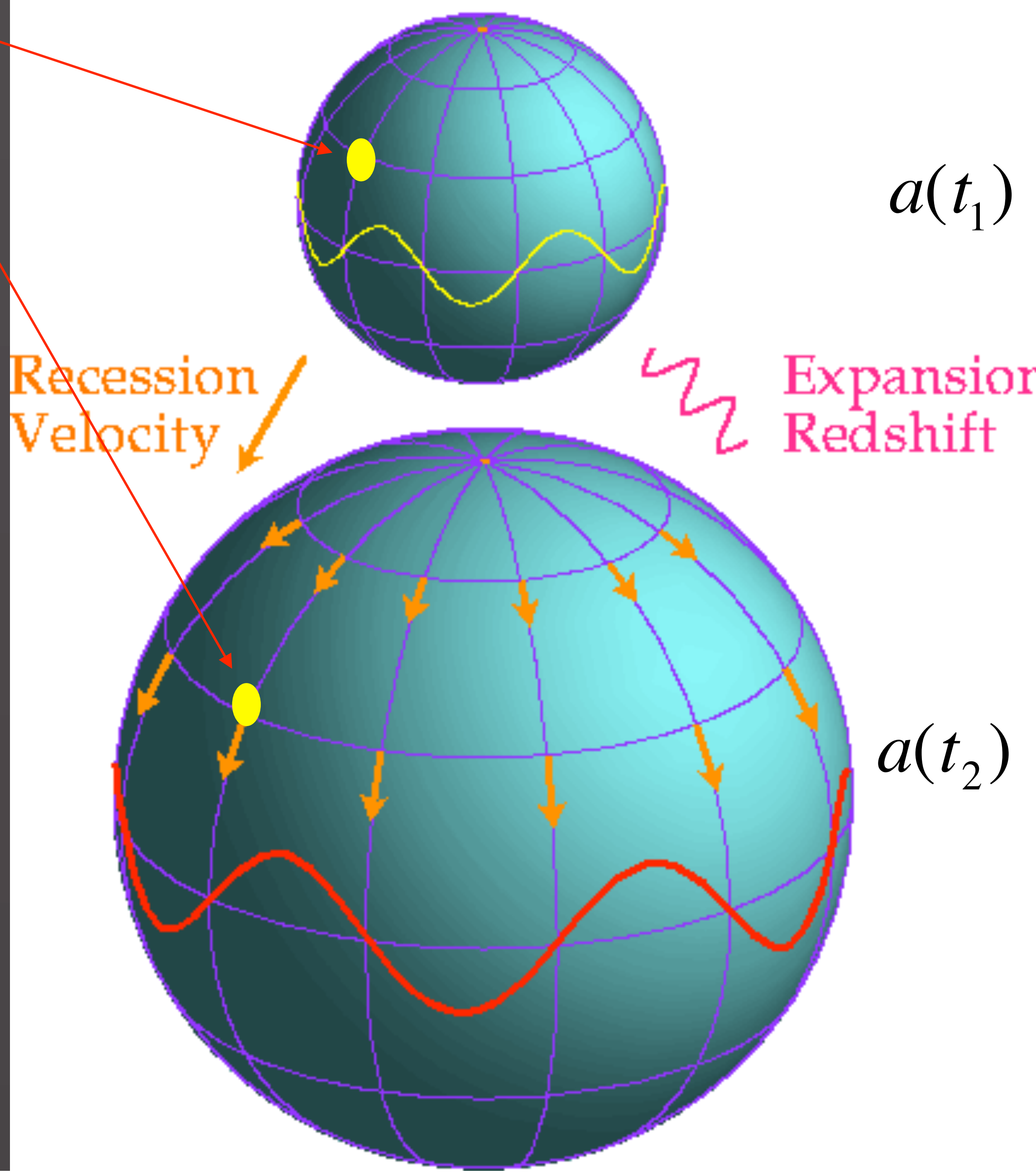
$$\lambda \sim a(t)$$

Redshift of light:

$$1 + z = \frac{\lambda(t_2)}{\lambda(t_1)} = \frac{a(t_2)}{a(t_1)}$$

indicates relative size of Universe directly

Cosmological Expansion



Cosmological Expansion

Distance between galaxies:

$$d(t) = a(t)r$$

where

$r =$ fixed comoving distance

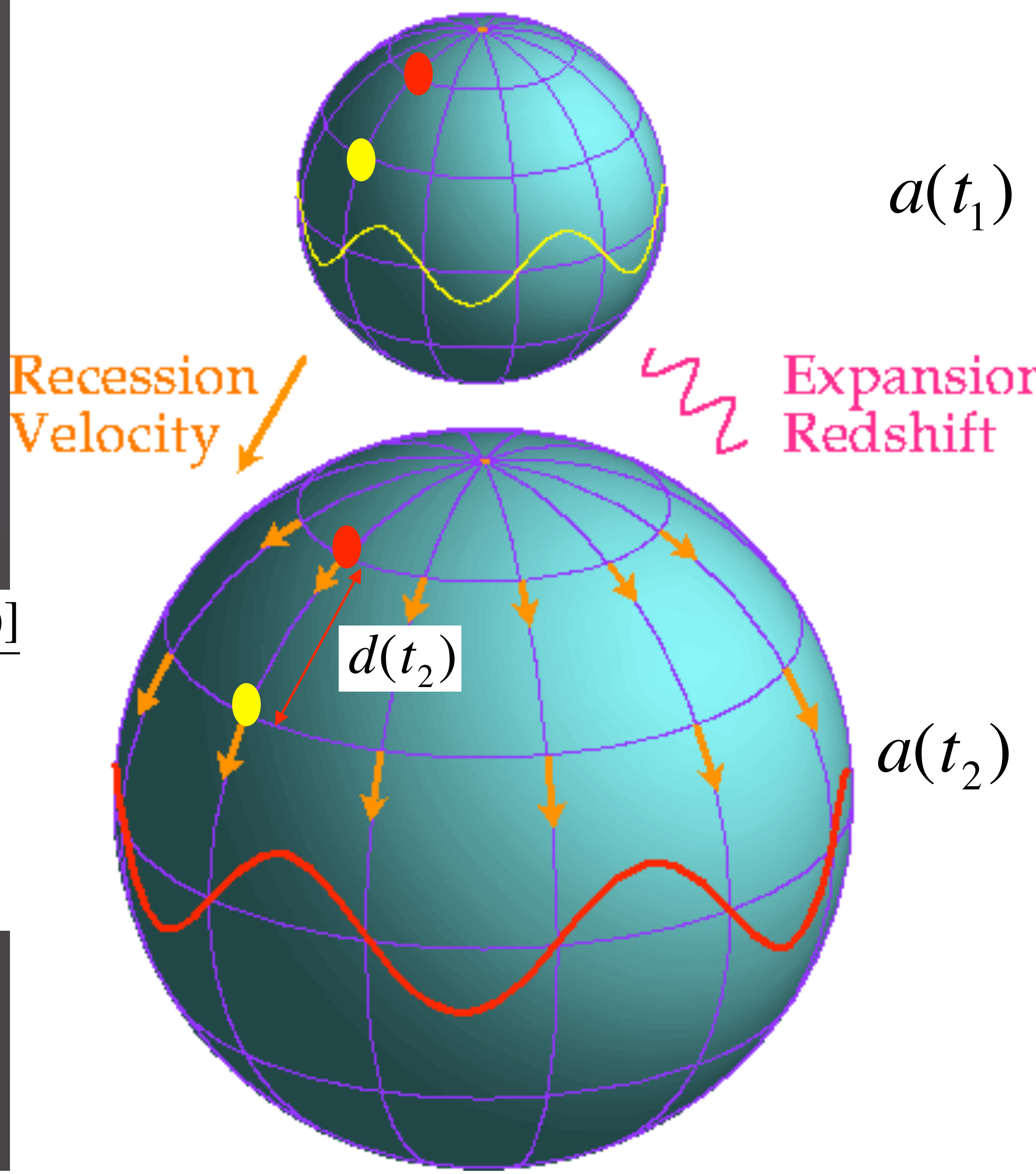
Recession speed:

$$\upsilon = \frac{d(t_2) - d(t_1)}{t_2 - t_1} = \frac{r[a(t_2) - a(t_1)]}{t_2 - t_1}$$

$$= \frac{d}{a}\frac{da}{dt} \equiv dH(t)$$

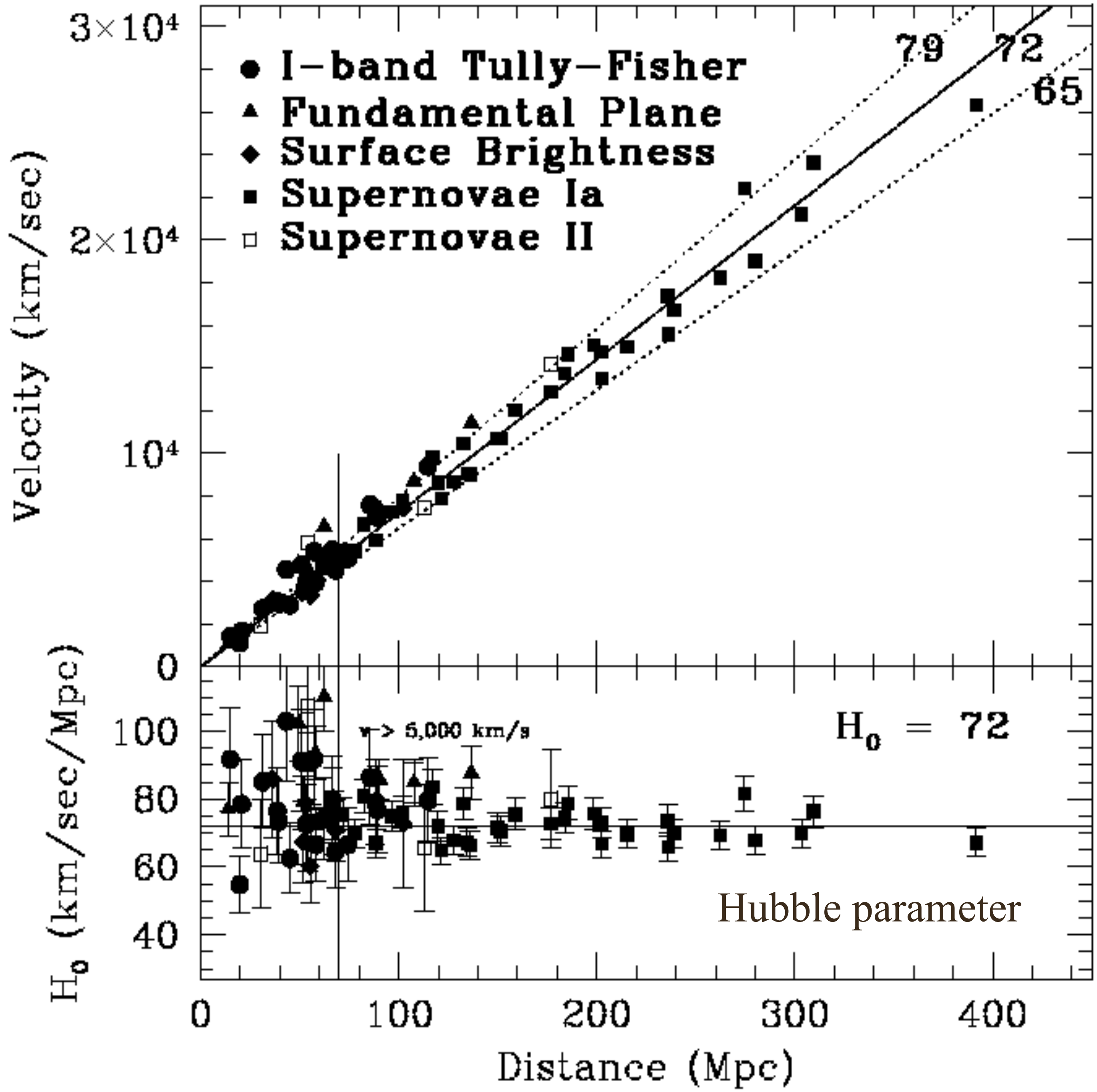
$$\approx dH_0 \text{ for `small' } t_2 - t_1$$

Hubble's Law (1929)



Modern Hubble Diagram

Hubble Space Telescope Key Project



Freedman etal

Expansion Kinematics

- Taylor expand about present epoch:

$$a(t) = a(t_0) + \dot{a}(t)|_0 (t - t_0) + \frac{1}{2}\ddot{a}(t)|_0 (t - t_0)^2 + ...$$

which implies to 2nd order in $t - t_0$:

$$\frac{a(t)}{a_0} = 1 + \left(\frac{\dot{a}}{a}\right)_0 (t - t_0) + \frac{1}{2}\left(\frac{\ddot{a}}{a}\right)_0 (t - t_0)^2 = 1 + H_0(t - t_0) - \frac{q_0 H_0^2}{2}(t - t_0)^2$$

where $H(t) = \dot{a}/a(t)$, $H_0 = (\dot{a}/a)\big|_{t=t_0}$ and $q_0 \equiv -(a\ddot{a}/\dot{a}^2)_0$

Differentiating with respect to t and keeping terms linear in $t - t_0$,

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}(t)}{a_0}\frac{a_0}{a(t)} = \left(H_0 - q_0 H_0^2 t + q_0 H_0^2 t_0\right)\left(1 - H_0(t - t_0)\right)$$
$$= H_0\left[1 - (1 + q_0)H_0(t - t_0)\right]$$

The redshift is given generally by

$$1 + z = \frac{a_0}{a(t)} \ ,$$

so that to this order of approximation from Eqn. 7 we have

$$z = -H_0(t - t_0) + \mathcal{O}(t - t_0)^2 \ ,$$

and we therefore find to this order,

$$H(z) = H_0\left[1 + (1 + q_0)z\right] \ .$$

Recent expansion history completely determined by H_0 and q_0

How does the expansion of the Universe change over time?

Gravity:

everything in the Universe attracts everything else

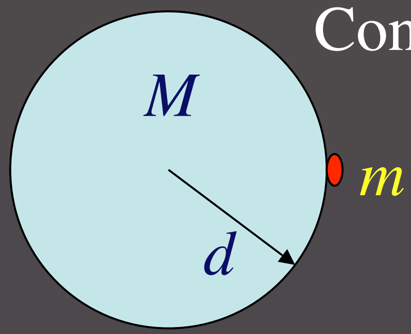
↓

expect the expansion of the Universe should slow down over time

Cosmological Dynamics

How does the scale factor of the Universe evolve?

Consider a homogenous ball of matter:



Kinetic Energy $m\upsilon^2/2$

Gravitational Energy $-GMm/d$

Conservation of Energy: $E = \frac{m\upsilon^2}{2} - \frac{GMm}{d}$ Birkhoff's theorem

Substitute $\upsilon = Hd$ and $M = (4\pi/3)\rho d^3$ to find

$$\frac{2E}{md^2} \equiv -\frac{K}{a^2(t)} = H^2(t) - \left(\frac{8\pi}{3}\right)G\rho(t)$$

1st order Friedmann equation

K interpreted as spatial curvature in General Relativity

Local Conservation of Energy-Momentum

First law of thermodynamics:

$$dE = -p dV$$

Energy:

$$E = \rho V \sim \rho a^3$$

First Law becomes:

$$\frac{d(\rho a^3)}{dt} = -p \frac{d(a^3)}{dt}$$

$$a^3 \dot{\rho} + 3\rho a^2 \dot{a} = -3 p a^2 \dot{a} \quad \Rightarrow$$

$$\frac{d\rho_i}{dt} + 3H(t)(p_i + \rho_i) = 0$$

2^{nd} Order Friedmann Equation

First order Friedmann equation:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k$$

Differentiate:

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(a^2\dot{\rho} + 2a\dot{a}\rho) \;\Rightarrow$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left[\dot{\rho}\left(\frac{a}{\dot{a}}\right) + 2\rho\right]$$

Now use conservation of energy - momentum:

$$\frac{d\rho_i}{dt} + 3H(t)(p_i + \rho_i) = 0 \;\Rightarrow$$

2nd order Friedmann equation:

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left[-3(p+\rho) + 2\rho\right] = -\frac{4\pi G}{3}\left[\rho + 3p\right]$$

Cosmological Dynamics

Spatial curvature: k=0,+1,-1

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i(t) - \frac{k}{a^2(t)}$$

Friedmann Equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i \left(\rho_i + \frac{3p_i}{c^2}\right)$$

Density Pressure

Equation of state parameter: $w_i = p_i / \rho_i c^2$

Non-relativistic matter: $p_m \sim \rho_m \mathrm{v}^2,\ w \approx 0$

Relativistic particles: $p_r = \rho_r c^2/3,\ w = 1/3$

The 2nd order Friedmann equation for a single component Universe gives

$$\left(\frac{\ddot{a}}{a}\right)_0 = -\frac{4\pi G}{3}(\rho_0 + 3p_0) \ . \tag{18}$$

From the first order Friedmann equation, the density parameter is given by

$$\Omega_0 = \frac{\rho_0}{\rho_{crit}} = \frac{\rho_0}{3H_0^2/8\pi G} = \frac{8\pi G\rho_0}{3H_0^2} \ , \tag{19}$$

so that

$$H_0^2 = \frac{8\pi G}{3}\frac{\rho_0}{\Omega_0} \ . \tag{20}$$

Combining Eqns. 18 and 20 gives the deceleration parameter,

$$\begin{aligned} q_0 &= -\left(\frac{a\ddot{a}}{\dot{a}^2}\right)_0 = -\frac{\ddot{a}_0}{H_0^2 a_0} = \frac{4\pi G}{3}(\rho_0 + 3p_0)\frac{3\Omega_0}{8\pi G\rho_0} \\ &= \frac{\Omega_0}{2}\left(1 + \frac{3p_0}{\rho_0}\right) \ . \end{aligned} \tag{21}$$

For a multi-component Universe, this generalizes to

$$q_0 = \frac{1}{2}\sum_i \Omega_i(1 + 3w_i) \ , \tag{22}$$

where the equation of state parameter $w_i = p_i/\rho_i$. For non-relativistic matter plus dark energy, this becomes

$$q_0 = \frac{\Omega_m}{2} + \frac{\Omega_{DE}}{2}(1 + 3w) \ . \tag{23}$$

Einstein-de Sitter Universe:
a special case

Non-relativistic matter: w=0, $\varrho_m \sim a^{-3}$

Spatially flat: $k=0 \rightarrow \Omega_m=1$

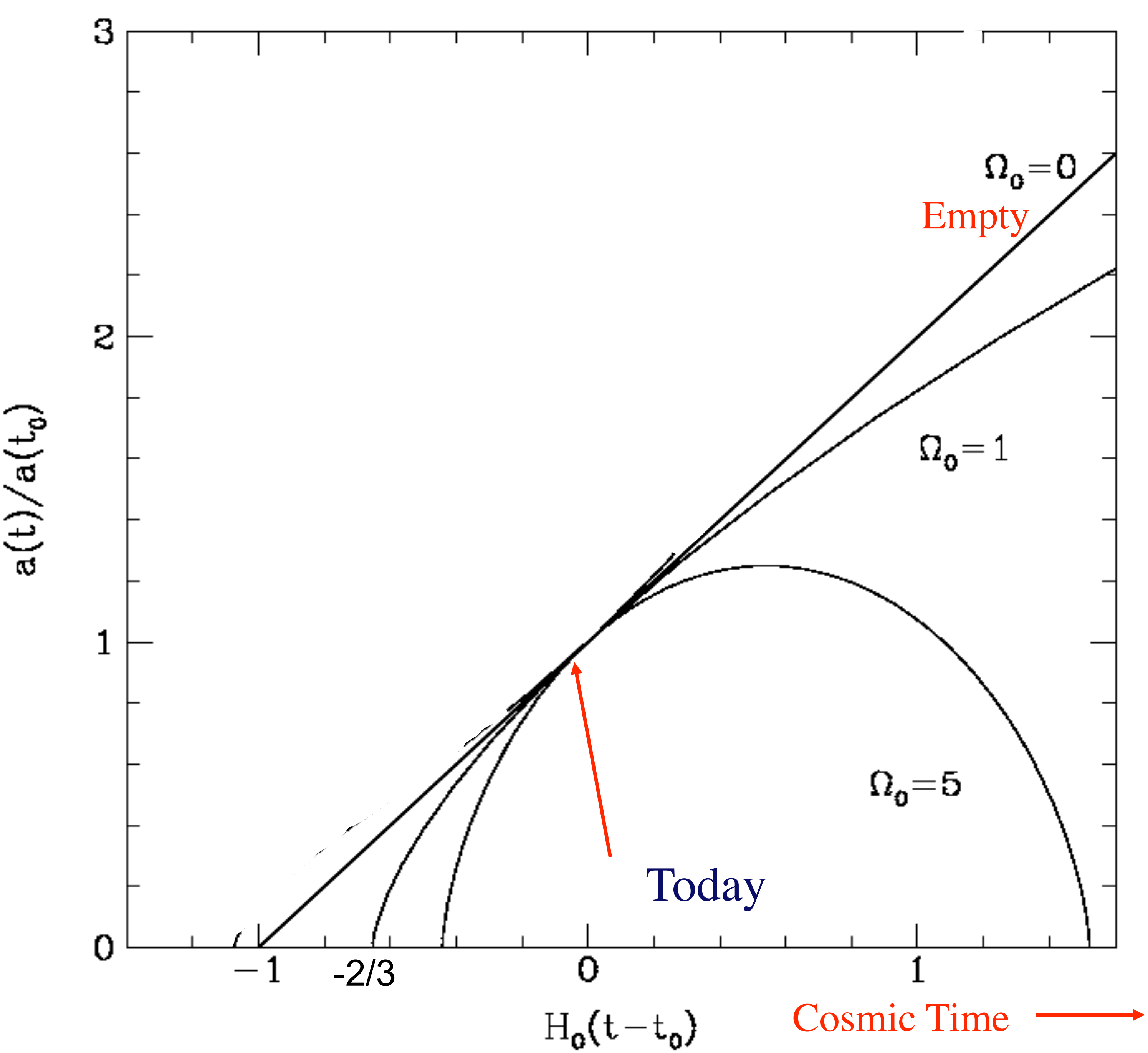
Friedmann:

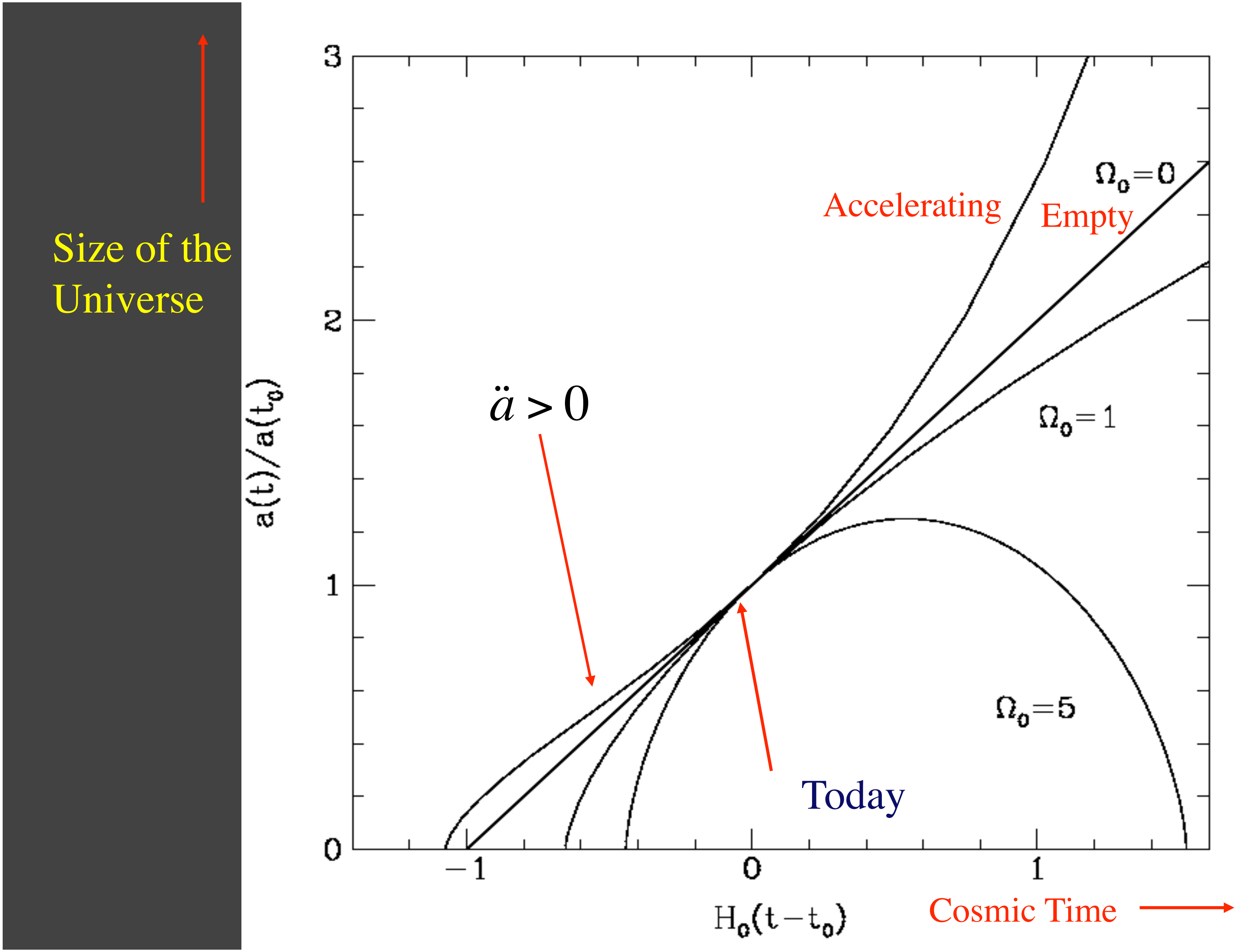
$$\left(\frac{\dot{a}}{a}\right)^2 \sim \frac{1}{a^3} \Rightarrow a^{1/2}da \sim dt \Rightarrow a \sim t^{2/3}$$

$$H = \frac{2}{3t}$$

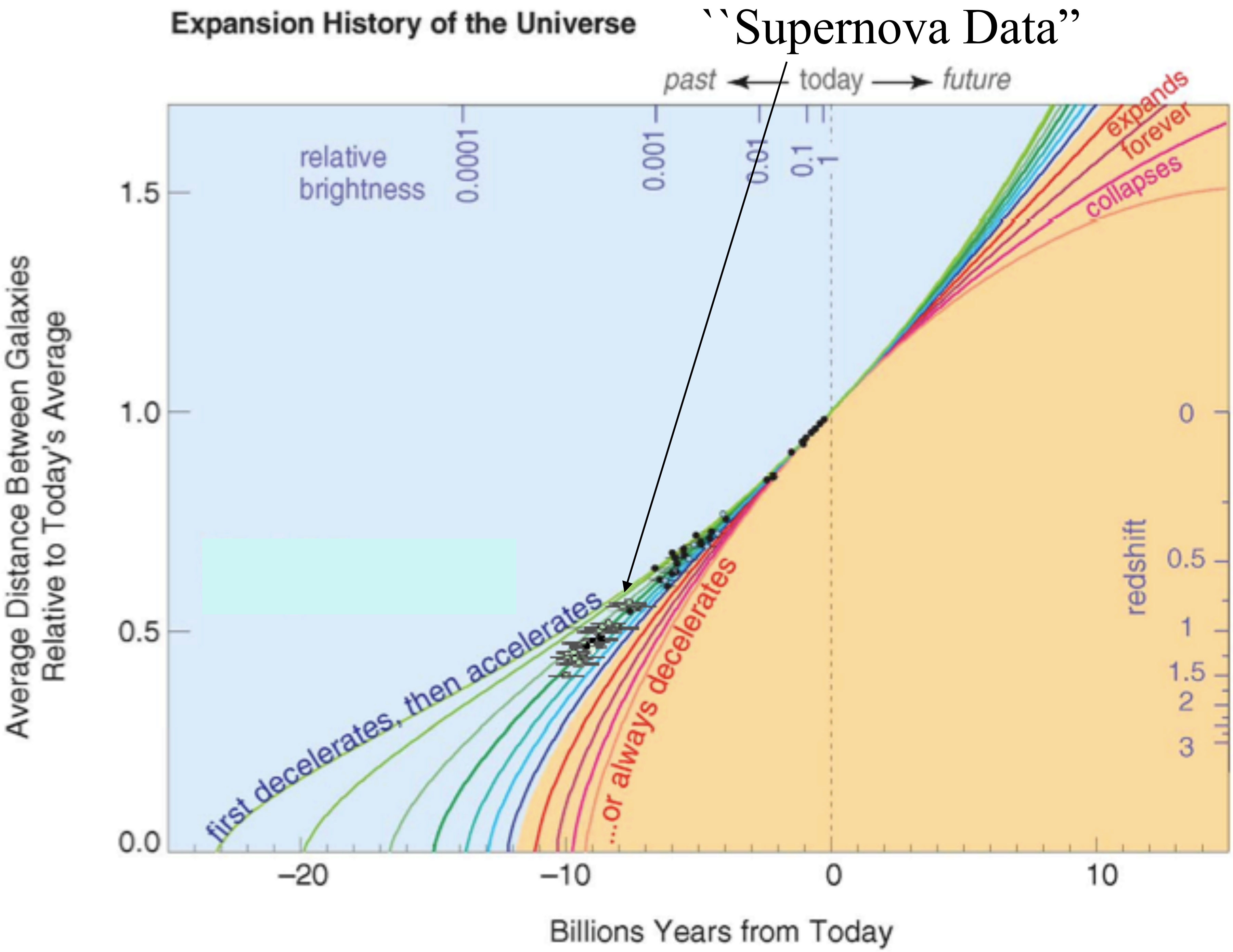
Size of the Universe

In these cases, $\dot{a}$ decreases with time, $\ddot{a} < 0$, expansion *decelerates* due to gravity



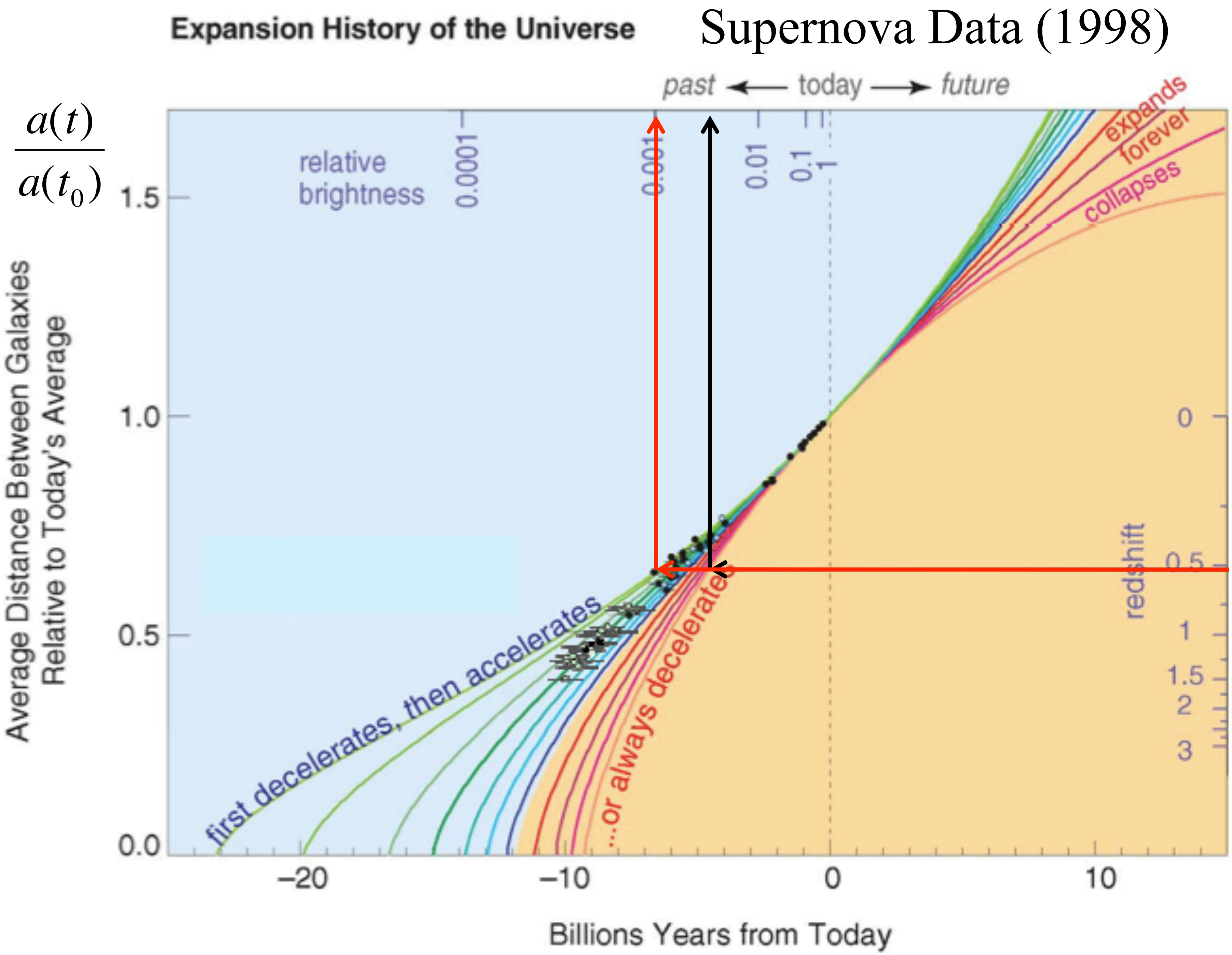


Size of the Universe
$a(t)/a(t_0)$
$\ddot{a} > 0$
Accelerating
Empty
$\Omega_0=0$
$\Omega_0=1$
$\Omega_0=5$
Today
$H_0(t-t_0)$
Cosmic Time



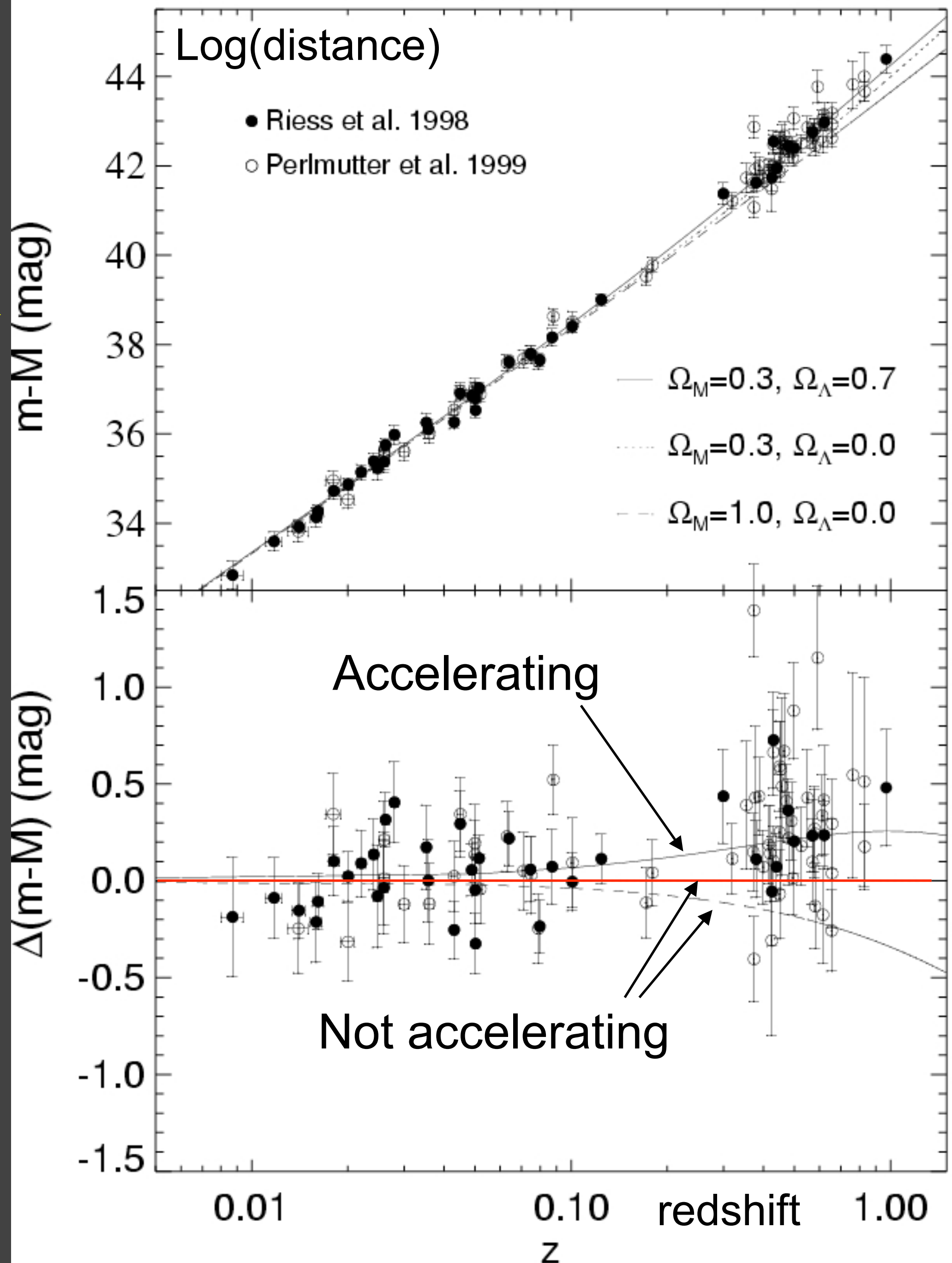
Expansion History of the Universe
``Supernova Data”
past
today
future
relative
brightness
0.0001
0.001
0.01
0.1
1
expands
forever
collapses
first decelerates, then accelerates
...or always decelerates
redshift
0
0.5
1
1.5
2
3
Average Distance Between Galaxies
Relative to Today's Average
1.5
1.0
0.5
0.0
−20
−10
0
10
Billions Years from Today

Supernova Data (1998)



Discovery of Cosmic Acceleration from High-redshift Supernovae

Type Ia supernovae that exploded when the Universe was 2/3 its present size are ~25% fainter than expected



$\Omega_\Lambda = 0.7$
$\Omega_\Lambda = 0.$
$\Omega_m = 1.$

Cosmological Dynamics

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i(t) - \frac{k}{a^2(t)}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i \left(\rho_i + \frac{3p_i}{c^2}\right)$$

Friedmann Equations

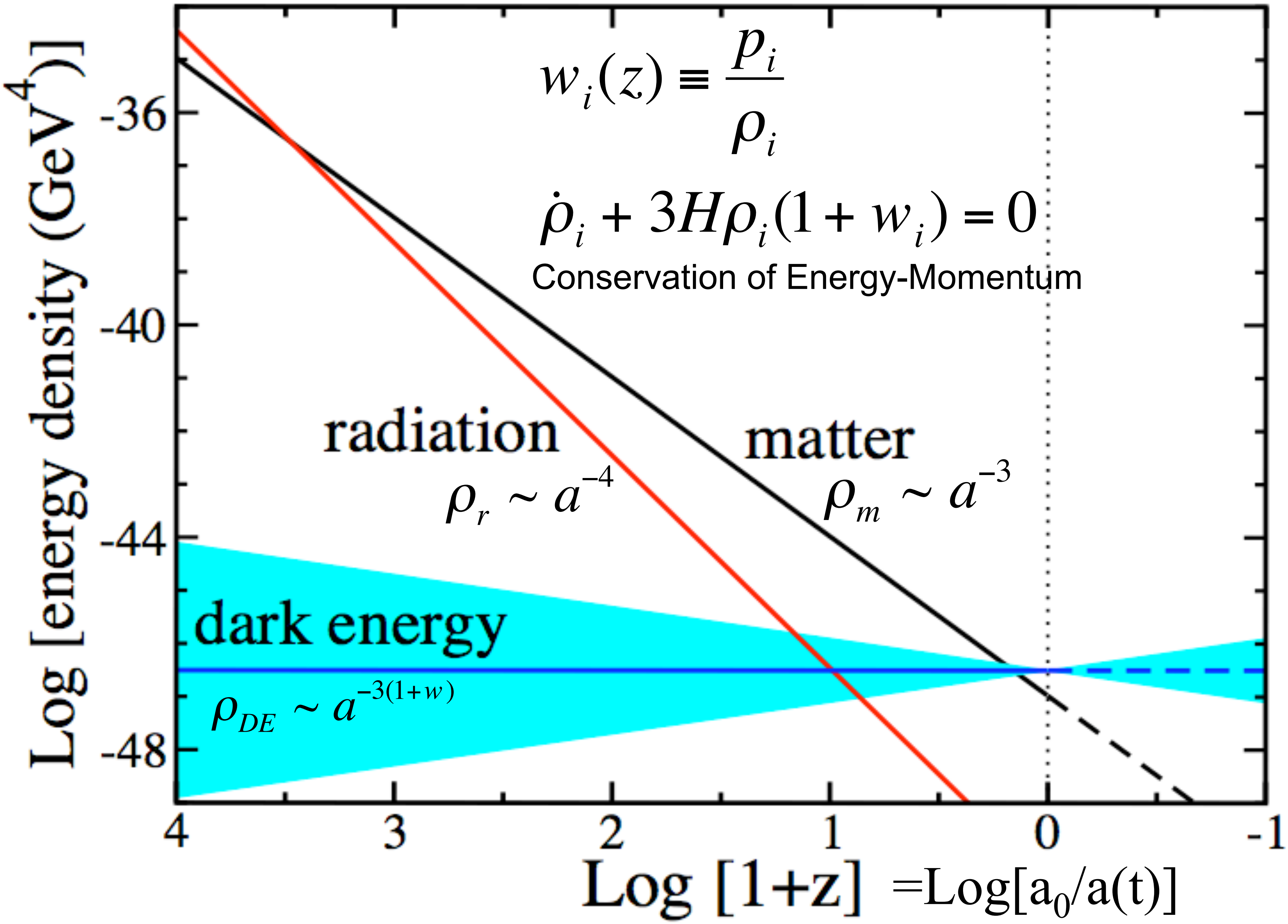
Equation of state parameter: $w_i = p_i / \rho_i c^2$

Non-relativistic matter: $p_m \sim \rho_m \mathrm{v}^2,\ w \approx 0$

Relativistic particles: $p_r = \rho_r c^2/3,\ w = 1/3$

Dark Energy: component with negative pressure: $w_{DE} < -1/3$

Equation of State parameter *w* determines Cosmic Evolution



Early 1990's: Circumstantial Evidence

The theory of primordial inflation successfully accounted for the large-scale smoothness of the Universe and the large-scale distribution of galaxies.

Inflation predicted what the total density of the Universe should be: the critical amount needed for the geometry of the Universe to be flat: $\Omega_{tot}=1$.

Measurements of the total amount of matter (mostly dark) in galaxies and clusters indicated not enough dark matter for a flat Universe ($\Omega_m=0.2$): there must be additional unseen stuff to make up the difference, if inflation is correct.

Measurements of large-scale structure (APM survey) were consistent with scale-invariant primordial perturbations from inflation with Cold Dark Matter plus Λ.

Cosmic Acceleration

$$\ddot{a} > 0 \rightarrow$$
$$\dot{a} = Ha \text{ increases with time}$$

This implies that $\upsilon = Hd$ increases with time: if we could watch the same galaxy over cosmic time, we would see its recession speed increase.

Exercise 1: A. Show that above statement is true.
B. For a galaxy at d=100 Mpc, if H_0=70 km/sec/Mpc =constant, what is the increase in its recession speed over a 10-year period? How feasible is it to measure that change?

What is the evidence for cosmic acceleration?

What could be causing cosmic acceleration?

How do we plan to find out?

Cosmic Acceleration

What can make the cosmic expansion speed up?

1. The Universe is filled with weird stuff that gives rise to `gravitational repulsion'. We call this

   Dark Energy

2. Einstein's theory of General Relativity is wrong on cosmic distance scales.

3. We must drop the assumption of homogeneity/isotropy.

Cosmological Constant as Dark Energy

Einstein:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Zel'dovich and Lemaitre:

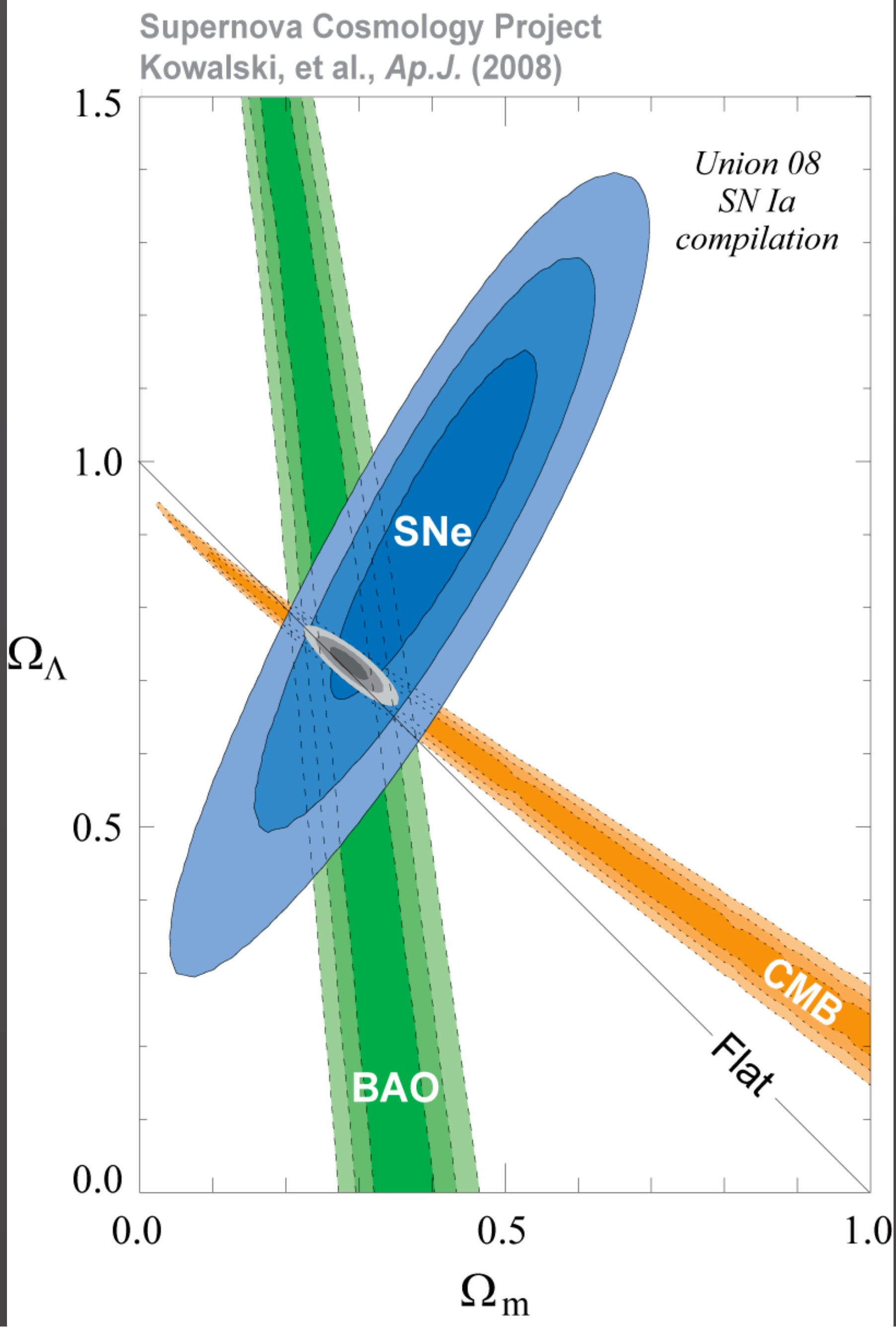
$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

$$\equiv 8\pi G\left(T_{\mu\nu}(\mathrm{matter}) + T_{\mu\nu}(\mathrm{vacuum})\right)$$

$$T_{\mu\nu}(\mathrm{vac}) = \frac{\Lambda}{8\pi \mathrm{G}} g_{\mu\nu}$$

$$\rho_{\mathrm{vac}} = T_{00} = \frac{\Lambda}{8\pi \mathrm{G}}, \quad p_{\mathrm{vac}} = T_{ii} = -\frac{\Lambda}{8\pi \mathrm{G}}$$

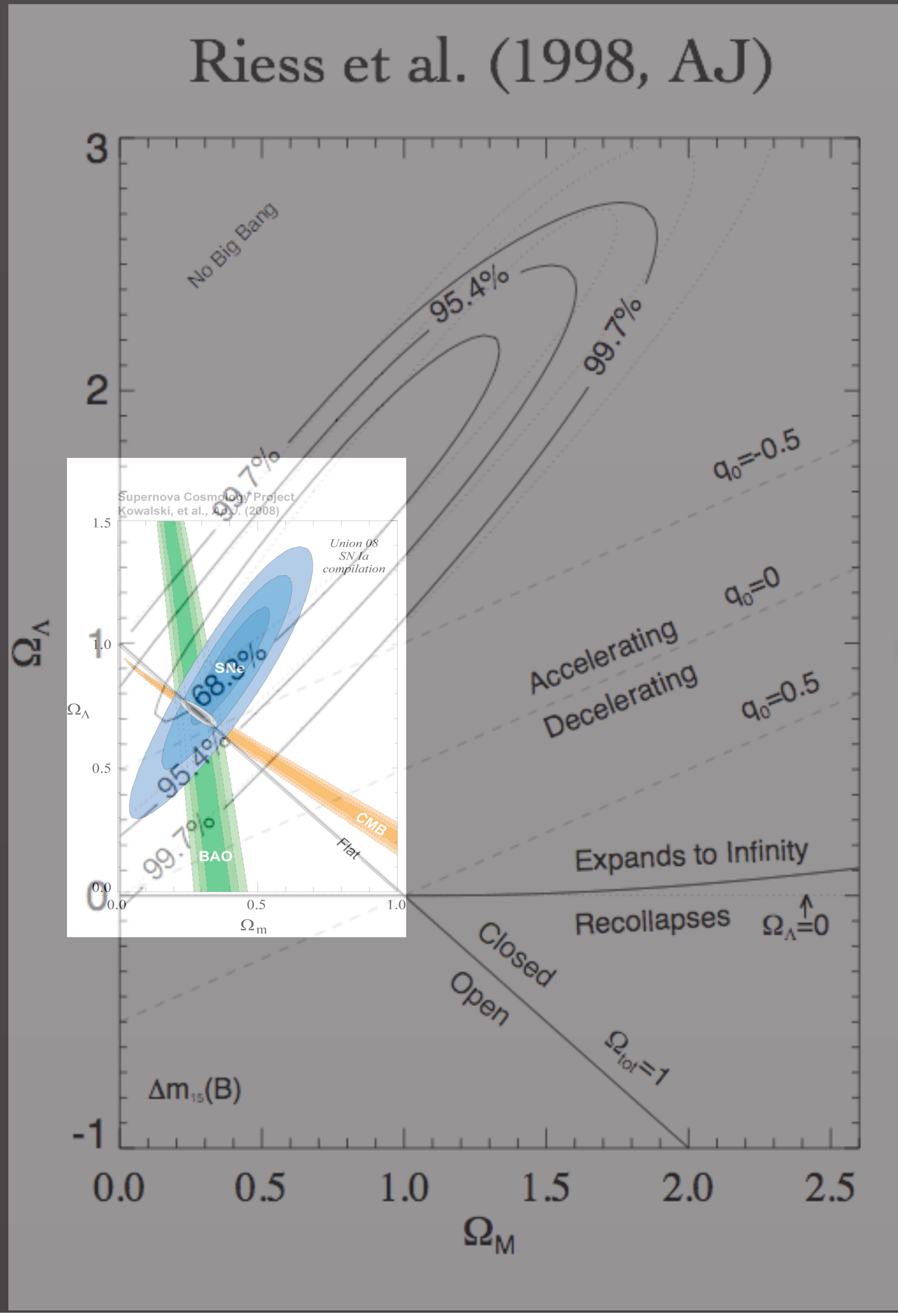
$$w_{\mathrm{vac}} = -1 \;\Rightarrow\; H = \mathrm{constant} \;\Rightarrow\; a(t) \propto \exp(Ht)$$



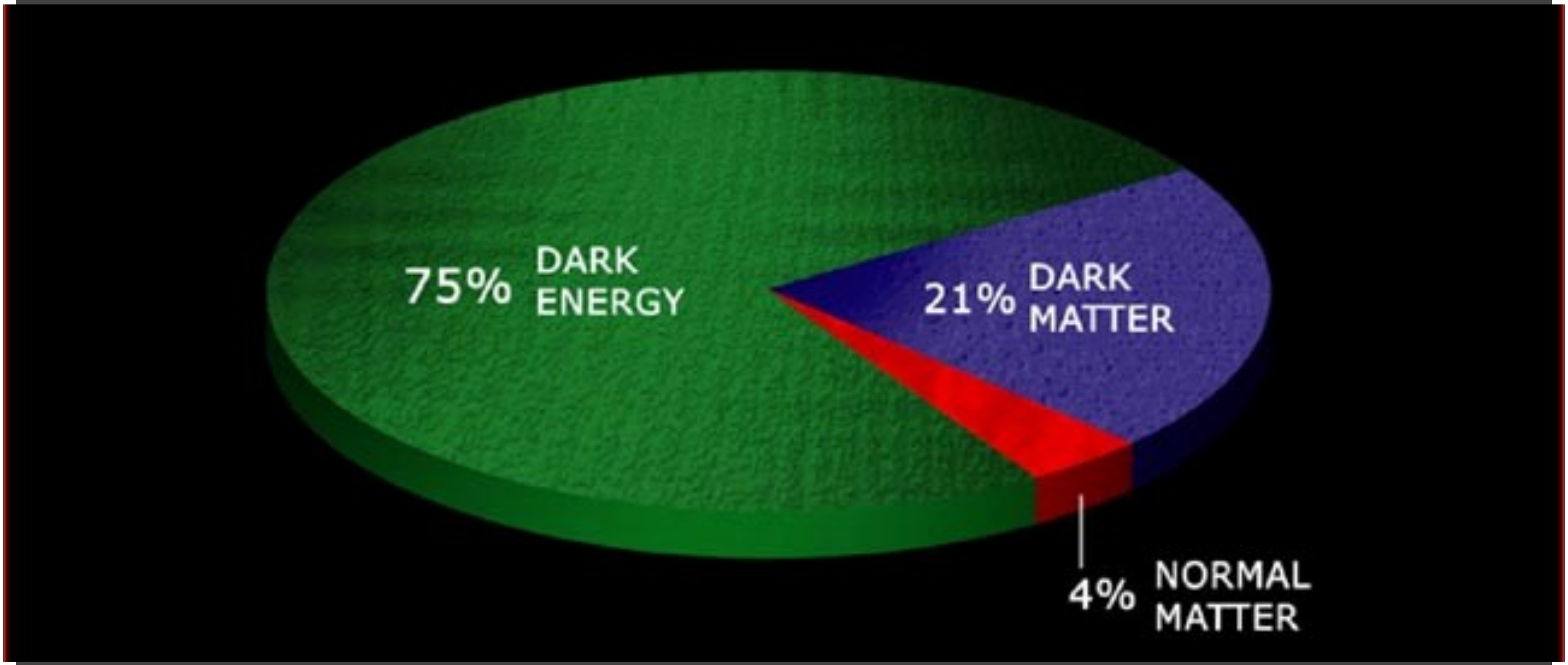
Recent Dark Energy Constraints

Constraints from Supernovae, Cosmic Microwave Background Anisotropy (WMAP) and Large-scale Structure (Baryon Acoustic Oscillations, SDSS)

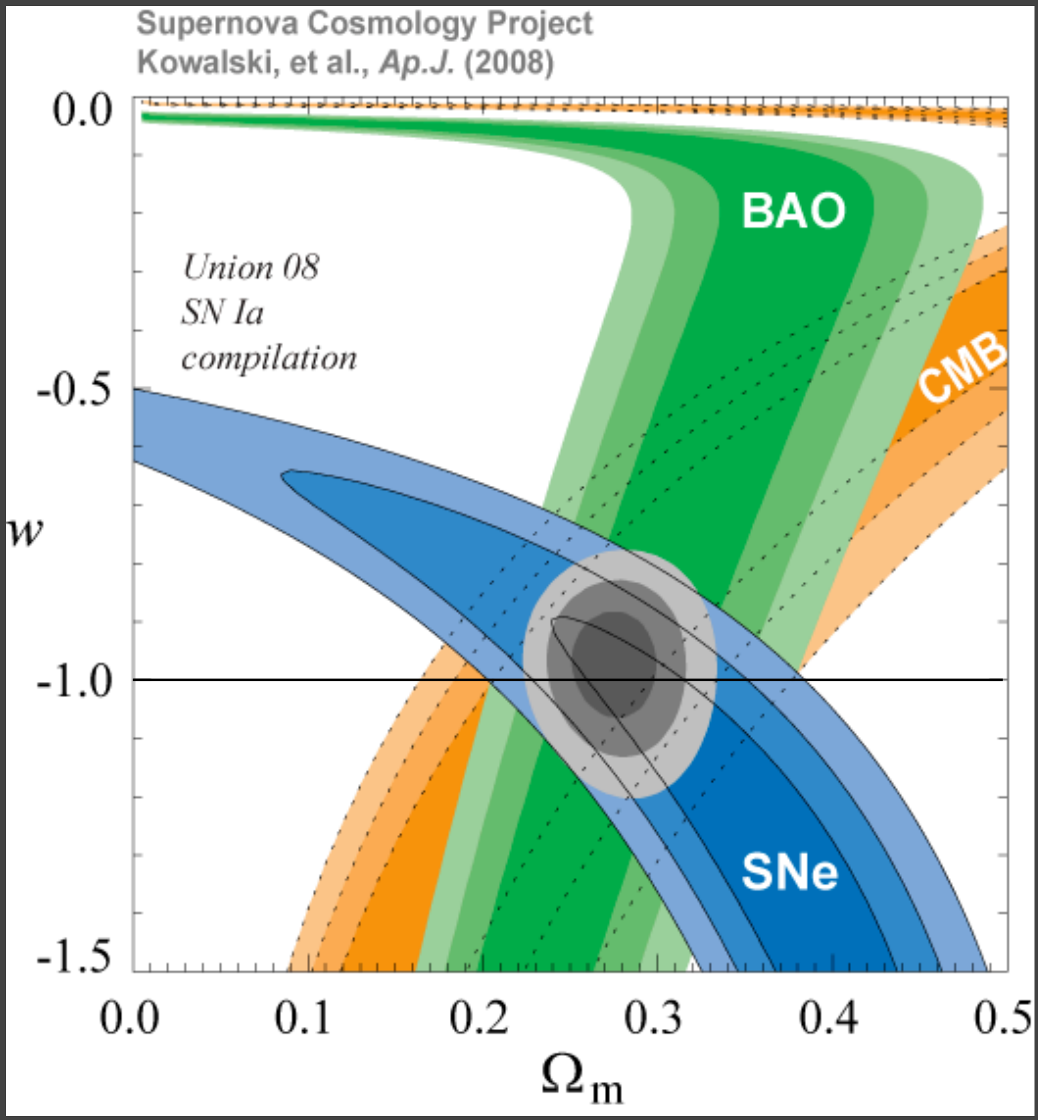
Progress over the last decade



Components of the Universe



Dark Matter: clumps, holds galaxies and clusters together
Dark Energy: smoothly distributed, causes expansion of Universe to speed up



assuming flat Univ. and constant *w*

Only statistical errors shown

History of Cosmic Expansion

- Depends on constituents of the Universe:

$$E^2(z) \equiv \frac{H^2(z)}{H_0^2} = \sum_i \Omega_i (1+z)^{3(1+w_i)} + \Omega_k (1+z)^2 \text{ for constant } w_i$$

$$= \Omega_m (1+z)^3 + \Omega_{DE} \exp\left[3\int (1+w(z))\, d\ln(1+z)\right] + \left(1-\Omega_m-\Omega_{DE}\right)\left(1+z\right)^2$$

where

$$\Omega_i = \frac{\rho_i}{\rho_{crit}} = \frac{\rho_i}{(3H_0^2/8\pi G)}$$

Cosmological Observables

Friedmann-Robertson-Walker Metric:

$$ds^2 = c^2dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)\left\{d\theta^2 + \sin^2\theta\, d\phi^2\right\}\right]$$
$$= c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left\{d\theta^2 + \sin^2\theta\, d\phi^2\right\}\right]$$

where $r = S_k(\chi) = \sinh(\chi),\ \chi,\ \sin(\chi)$ for $k = -1,0,1$

Comoving distance: $cdt = a\, d\chi \;\Rightarrow\; \chi = \int \frac{cdt}{a} = \int \frac{cdt}{ada} da = c\int \frac{da}{a^2 H(a)}$

$$a = \frac{1}{1+z} \;\Rightarrow\; da = -(1+z)^{-2}dz = -a^2dz$$

$$c\frac{dt}{da}da = ad\chi \;\Rightarrow\; -\frac{c}{\dot{a}}a^2dz = ad\chi \;\Rightarrow\; -cdz = H(z)d\chi$$

Age of the Universe

$$cdt = ad\chi$$

$$t = \int a d\chi = \int \frac{da}{aH(a)} = \int \frac{dz}{(1+z)H(z)}$$

$$t_0 = \frac{1}{H_0} \int_0^{\infty} \frac{dz}{(1+z)E(z)}$$

where $E(z) = H(z)/H_0$

Exercise 2:

$$E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_m (1+z)^3 + \Omega_{DE} \exp\left[3\int (1+w(z))\, d\ln(1+z)\right] + \left(1 - \Omega_m - \Omega_{DE}\right)(1+z)^2$$

A. For $w=-1$ (cosmological constant Λ) and $k=0$:

$$E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_m a^{-3} + \Omega_\Lambda$$

Derive an analytic expression for $H_0 t_0$ in terms of Ω_m

Plot $H_0 t_0$ vs. Ω_m

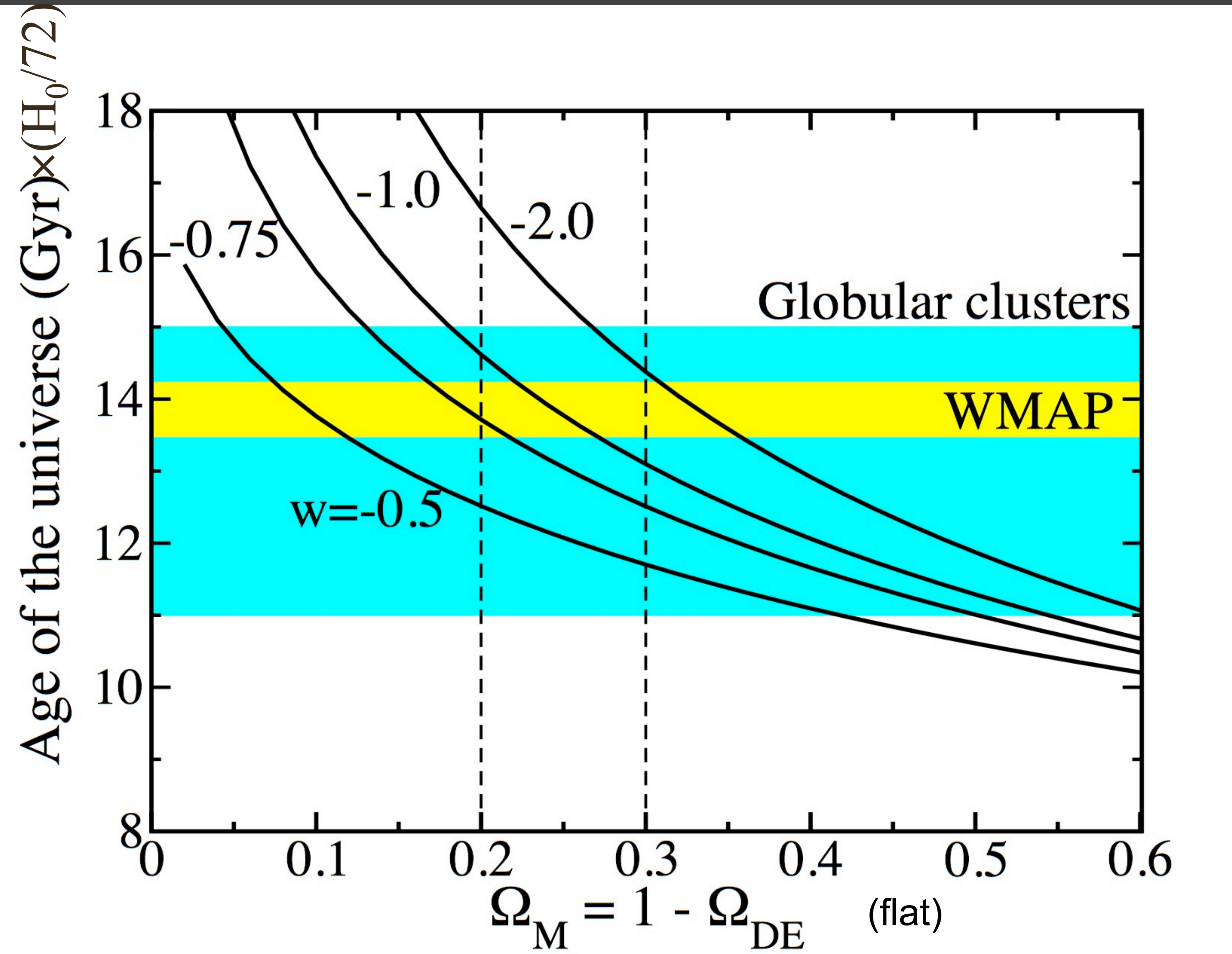
B. Do the same, but for $\Omega_\Lambda = 0,\ \Omega_k \neq 0$

C. Suppose H_0=70 km/sec/Mpc and t_0=13.7 Gyr.

Determine Ω_m in the 2 cases above.

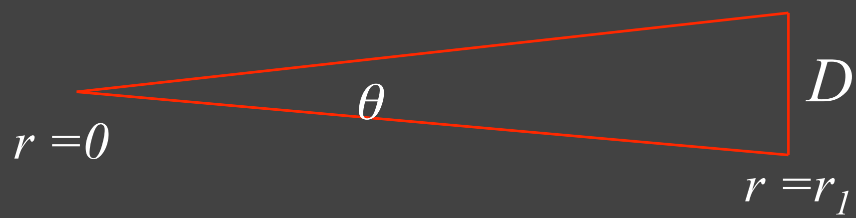
D. Repeat part C but with H_0=72.

Age of the Universe



Angular Diameter Distance

- Observer at $r = 0$, t_0 sees source of proper diameter D at coordinate distance $r = r_1$ which emitted light at $t = t_1$:

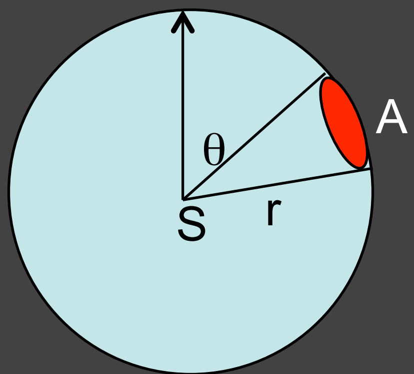


- From FRW metric, proper distance across the source is $D = a(t_1) r_1 \theta$ so the angular diameter of the source is $\theta = D / a_1 r_1$
- In Euclidean geometry, $d = D/\theta$ so we define the

Angular Diameter Distance: $$d_A \equiv \frac{D}{\theta} = a_1 r_1 = a_1 S_k(\chi_1) = \frac{r_1 a_0}{1 + z_1}$$

Luminosity Distance

- Source *S* at origin emits light at time t_1 into solid angle $d\Omega$, received by observer *O* at coordinate distance *r* at time t_0, with detector of area *A*:



Proper area of detector given by the metric:

$$A = a_0 r\, d\theta\, a_0 r \sin\theta\, d\phi = a_0^2 r^2 d\Omega$$

Unit area detector at *O* subtends solid angle

$d\Omega = 1/a_0^2 r^2$ at *S*.

Power emitted into $d\Omega$ is $dP = L\, d\Omega / 4\pi$

Energy flux received by *O* per unit area is

$$f = \frac{L\, d\Omega}{4\pi} = \frac{L}{4\pi a_0^2 r^2}$$

Include Expansion

- Expansion reduces received flux due to 2 effects:
  1. Photon energy redshifts: $E_\gamma(t_0) = E_\gamma(t_1)/(1+z)$
  2. Photons emitted at time intervals δt_1 arrive at time intervals δt_0:

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)}$$

$$\int_{t_1}^{t_1+\delta t_1} \frac{dt}{a(t)} + \int_{t_1+\delta t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)}$$

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)} \quad \Rightarrow \quad \frac{\delta t_0}{\delta t_1} = \frac{a(t_0)}{a(t_1)} = 1+z$$

$$f = \frac{L\, d\Omega}{4\pi} = \frac{L}{4\pi a_0^2 r^2 (1+z)^2} \equiv \frac{L}{4\pi d_L^2} \quad \Rightarrow \quad d_L = a_0 r(1+z) = (1+z)^2 d_A$$

Convention: choose $a_0=1$

Luminosity Distance

Worked Example I

$$E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_m (1+z)^3 + \Omega_{DE} \exp\left[3\int (1+w(z))\, d\ln(1+z)\right] + \left(1-\Omega_m-\Omega_{DE}\right)(1+z)^2$$

For *w=-1*(cosmological constant Λ):

$$E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_m a^{-3} + \Omega_\Lambda + \left(1-\Omega_m-\Omega_\Lambda\right)a^{-2}$$

Luminosity distance:

$$d_L(z;\Omega_m,\Omega_\Lambda) = r(1+z) = c(1+z)S_k\left(\int \frac{da}{H_0 a^2 E(a)}\right)$$

$$= c(1+z)S_k\left(\int \frac{da}{H_0 a^2[\Omega_m a^{-3} + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)a^{-2}]^{1/2}}\right)$$

Worked Example II

$$E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_{DE}\exp\left[3\int(1+w(z))\,d\ln(1+z)\right] + \left(1-\Omega_m-\Omega_{DE}\right)(1+z)^2$$

For a flat Universe (*k=0*) and constant Dark Energy equation of state *w*:

$$E^2(z) = \frac{H^2(z)}{H_0^2} = (1-\Omega_{DE})(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}$$

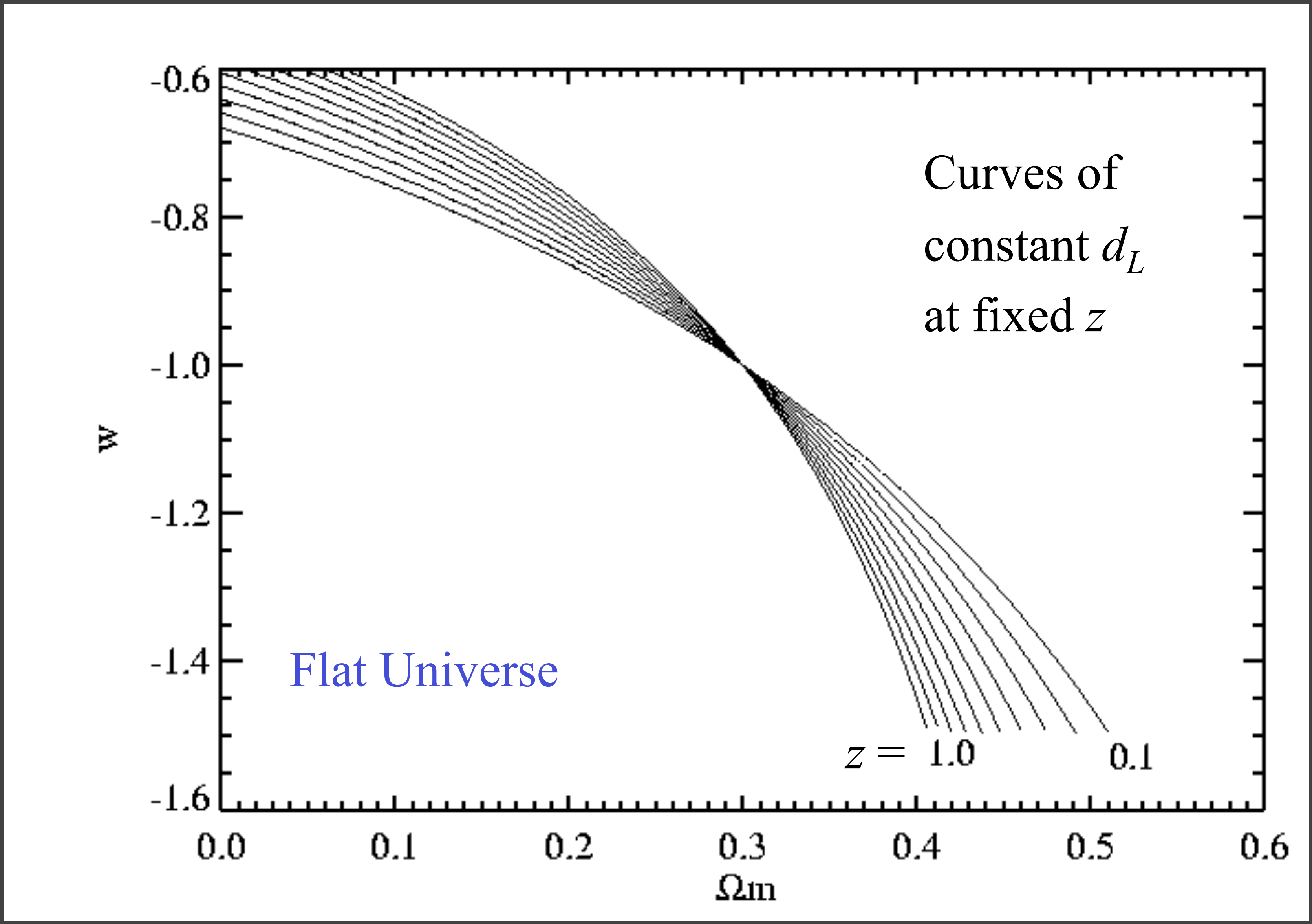
Luminosity distance:

$$d_L(z;\Omega_{DE},w) = r(1+z) = \chi(1+z) = \frac{c(1+z)}{H_0}\int\frac{da}{a^2E(a)}$$

$$= \frac{c(1+z)}{H_0}\int\frac{1+\Omega_{DE}[(1+z)^{3w}-1]^{-1/2}}{(1+z)^{3/2}}dz$$

Note: $H_0 d_L$ is independent of H_0

Dark Energy Equation of State



Exercise 3

- Make the same plot for Worked Example I: plot curves of constant luminosity distance (for several choices of redshift between 0.1 and 1.0) in the plane of $\Omega_\Lambda \text{ vs. } \Omega_m$, choosing the distance for the model with $\Omega_\Lambda = 0.7,\ \Omega_m = 0.3$ as the fiducial.

- In the same plane, plot the boundary of the region between *present* acceleration and deceleration.
- Extra credit: in the same plane, plot the boundary of the region that expands forever vs. recollapses.

Bolometric Distance Modulus

- Logarithmic measures of luminosity and flux:

$$M = -2.5\log(L) + c_1, \quad m = -2.5\log(f) + c_2$$

- Define distance modulus:

flux — measure redshift from spectra

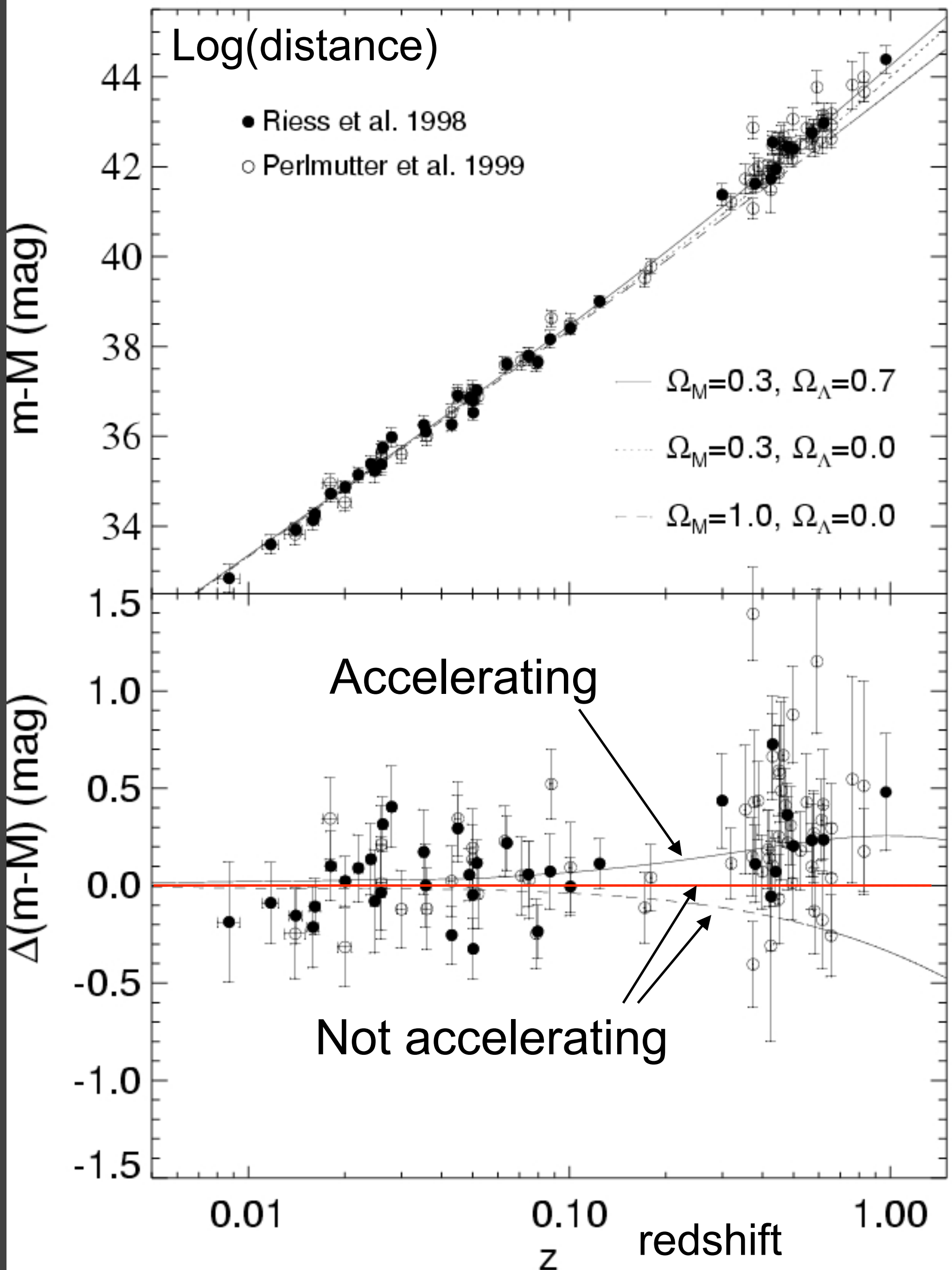
$$\mu \equiv m - M = 2.5\log(L/f) + c_3 = 2.5\log(4\pi d_L^2) + c_3$$
$$= 5\log[H_0 d_L(z;\Omega_m,\Omega_{DE},w(z))] - 5\log H_0 + c_4$$
$$= 5\log[d_L(z;\Omega_m,\Omega_{DE},w(z))/10\text{pc}]$$

- For a population of *standard candles* (fixed *M*), measurements of μ vs. z, the Hubble diagram, constrain cosmological parameters.

Exercise 4

- Plot distance modulus vs redshift (z=0-1) for:
    - Flat model with $\Omega_m = 1$
    - Flat model with $\Omega_\Lambda = 0.7,\ \Omega_m = 0.3$
    - Open model with $\Omega_m = 0.3$
  - Plot first linear in z, then log z.
- Plot the residual of the first two models with respect to the third model

Discovery of Cosmic Acceleration from High-redshift Supernovae



Type Ia supernovae that exploded when the Universe was 2/3 its present size are ~25% fainter than expected

$\Omega_\Lambda = 0.7$
$\Omega_\Lambda = 0.$
$\Omega_m = 1.$

Distance and q_0

$$H(z) = H_0\left[1 + (1 + q_0)z\right] \ . \tag{11}$$

The coordinate distance is

$$a_0\chi = a_0 \int \frac{dt}{a(t)} = a_0 \int \frac{dt}{da}\frac{da}{a} = a_0 \int \frac{da}{H(a)a^2} \ . \tag{12}$$

Using Eqn. 9, this can be written as

$$a_0\chi(z) = \int \frac{dz}{H(z)} \ . \tag{13}$$

Using Eqn. 11, this becomes

$$a_0\chi(z) = \int \frac{dz}{H_0[1 + (1 + q_0)z]} \simeq \frac{1}{H_0} \int dz[1 - (1 + q_0)z] = \frac{1}{H_0}\left[z - (1 + q_0)\frac{z^2}{2}\right] \ . \tag{14}$$

The radial distance $r = \sin\chi, \chi, \sinh\chi$ for $k = +1, 0, -1$. For small distances, $\chi \ll 1$, this means $r = \chi \pm \mathcal{O}(\chi^3)$. Since, from Eqn. 14, $\chi \propto z + \mathcal{O}(z^2)$, the expression for $a_0 r(z)$ to $\mathcal{O}(z^2)$ is identical to the expression for $a_0\chi(z)$ to the same order, i.e., Eqn. 14.

Distance and q_0

The luminosity distance is given by $d_L(z) = (1+z)a_0 r(z)$. Using Eqn. 14 and the result of part (d), to order z^2 this gives

$$\begin{aligned} d_L(z; H_0, q_0) &= \frac{z(1+z)}{H_0}\left[1 - (1+q_0)\frac{z}{2}\right] = \frac{1}{H_0}\left[z + z^2 - (1+q_0)\frac{z^2}{2} + \mathcal{O}(z^3)\right] \\ &= \frac{z}{H_0}\left[1 + (1-q_0)\frac{z}{2}\right] \; . \end{aligned} \tag{15}$$

The distance modulus is given by

$$\begin{aligned} \mu(z; H_0, q_0) &= 5\log_{10}\left(\frac{d_L}{10\ \mathrm{pc}}\right) = 5\log_{10}\left[\frac{z}{H_0}\frac{1+(1-q_0)z/2}{10\ \mathrm{pc}}\right] \\ &= 5\log z - 5\log(H_0 \cdot 10\ \mathrm{pc}) + 5\log\left[1 + \frac{z}{2}(1-q_0)\right] \; . \end{aligned} \tag{16}$$

The last term in Eqn. 16 can be massaged using Stirling's approximation: for $x \ll 1$, $\ln(1+x) \simeq x$. Exponentiating and taking the $\log_{10}$ gives $\log_{10}(1+x) \simeq \log_{10} e^x = x\log_{10} e$, so that

$$5\log_{10}\left[1 + \frac{z}{2}(1-q_0)\right] \simeq \frac{5z}{2}(1-q_0)\log_{10} e = 1.086z(1-q_0) \; . \tag{17}$$

Recall $q_0 = \dfrac{\Omega_m}{2} + \dfrac{\Omega_{DE}}{2}(1+3w)$

Distance and q_0

Recall $q_0 = \frac{\Omega_m}{2} + \frac{\Omega_{DE}}{2}(1+3w)$

For a flat Universe, $\Omega_{DE} = 1 - \Omega_m$; from Eqn. 22,

$$q_0 = \frac{\Omega_m}{2} + \frac{(1-\Omega_m)}{2}(1+3w) = \frac{1}{2} + \frac{3w}{2}(1-\Omega_m) \ , \tag{25}$$

so the difference in distance modulus between two flat models with fixed H_0 and Ω_m is

$$\Delta\mu = \frac{3}{2}(1-\Omega_m)(1.086z)\Delta w = 0.6\Delta w \ , \tag{26}$$

where the last expression is evaluated using $\Omega_m = 0.25$ and $z = 0.5$. Since $\sigma_\mu = 0.15$ mag, to determine w to a precision of $\Delta w = 0.1$ requires roughly $\Delta\mu = 0.06 > \sigma_\mu/\sqrt{N} = 0.15/\sqrt{N}$, or $N > 6$ supernovae. For a precision $\Delta w = 0.01$, we have $\Delta\mu = 0.006$, and we need $N > 600$ supernovae at $z \sim 0.5$. If Ω_m isn't exactly known and in the presence of systematic errors, this number of course would be larger.