Astronomy 304: Astrophysics IV
Spring Quarter 2010

Problem Set 1

Due: Tuesday, April 20

If you have any questions, please contact me *before* the due date by email: frieman@fnal.gov.

1. Suppose the Universe consists of non-relativistic, pressureless matter (m) with present density parameter $\Omega_m = \rho_m(t_0)/\rho_{crit}(t_0)$, radiation (relativistic particles) with present density paramter Ω_r, and a dark energy component (d) with equation of state $p_d = w_d \rho_d$, where $w_d < 0$ is constant, and present density parameter Ω_d.
   (a) Derive an expression for the deceleration parameter $q_0 \equiv -(\ddot{a}/H^2 a)_0$ in terms of these density parameters and w_d.
   (b) For the rest of this problem, neglect the tiny present radiation density, i.e., set $\Omega_r = 0$. Observations of Type Ia supernovae indicate that the Universe is currently accelerating. Suppose the dark energy component is a cosmological constant, i.e., that $w_d = -1$, and that the Universe is spatially flat ($k = 0$, i.e., $\Omega_{total} = 1$). Find an upper bound on Ω_m such that the Universe could now be accelerating.
   (c) Now relax the assumption above on the dark energy equation of state parameter but retain the assumption of spatial flatness. In a plot of w_d vs. Ω_m, sketch the boundary of the region $q_0 = 0$. Now assume that observations of galaxy clusters (or other measurements) indicate $\Omega_m \geq 0.2$: what bound does this place on w_d such that present accelerated expansion is possible?

2. Consider a Friedmann Universe with non-relativistic matter and a cosmological constant.
   (a) Assuming spatial flatness ($k = 0$, i.e., $\Omega_m + \Omega_\Lambda = 1$), derive an exact analytic expression for the quantity $H_0 t_0$ purely in terms of Ω_Λ (where we define $\Omega_\Lambda = \Omega_d(w_d = -1)$). Plot this quantity as a function of Ω_m.
   (b) Now consider a Universe with non-relativistic matter, *zero* cosmological constant, and assume $\Omega_m < 1$, i.e., an open universe. Derive an analytic expression for the same quantity $H_0 t_0$ in terms of Ω_m for this case and plot the result on the same graph as above.
   (c) Suppose $H_0 = 70$ km/sec/Mpc and the age of the Universe $t_0 = 13.7$ billion years. Estimate the corresponding values for Ω_m in the two cases above. Repeat the calculation assuming $H_0 = 72$ and $t_0 = 13.7$ Gyr.

3. Consider a homogeneous and isotropic Universe filled with non-relativistic matter and a cosmological constant Λ, where $p_\Lambda = -\rho_\Lambda = -\Lambda/8\pi G$. Einstein originally introduced the cosmological constant into General Relativity because he thought the Universe was static (not contracting or expanding).

   (a) In terms of the matter density ρ_m, find the critical value of the cosmological constant, denoted by Λ_c, such that the Universe is static.

   (b) What is the spatial geometry of this static model, known as the Einstein static universe?

   (c) By perturbing the Friedmann equations around the static solution (but preserving homogeneity and isotropy), show that the Einstein static universe is globally unstable. In terms of Λ_c, what is the characteristic growth time for the instability?

4. Consider an expanding homogeneous and isotropic Universe with non-relativistic matter satisfying $\Omega_m > 1$ today. In the absence of a cosmological constant, we know that in such a model the expansion will halt and the Universe will eventually recollapse. Now add a cosmological constant to this model, with present density parameter Ω_Λ. Show that if Ω_Λ exceeds a critical value, the Universe will avoid recollapse, and express this critical value analytically in terms of Ω_m. Plot the boundary region between eventual recollapse and continued expansion in the Ω_m (x-axis) vs. Ω_Λ (y-axis) plane, for values of Ω_m between 0 and 3. For comparison, also plot the line $\Omega_\Lambda = 0$.

5. For a Friedmann Universe containing only non-relativistic matter, with present density parameter $\Omega_m(0)$, derive an expression for the density parameter as a function of redshift z, i.e., express $\Omega_m(z)$ analytically in terms of $\Omega_m(0)$ and z. Also express the expansion rate $H(z)$ analytically in terms of H_0, $\Omega_m(0)$, and z. Now suppose $\Omega_m(0) = 0.2$: find the redshift z_m at which $\Omega_m(z_m) = 0.5$, i.e., at which the Universe became curvature-dominated (roughly). For more recent epochs, such that $z \ll z_m$, what is the approximate solution for the cosmic scale factor?