(1) Transit Light curve

The first step is to figure out what the orbital period and phase are. We have Winn’s equation 32 as: \( t_c[n] = t_c[0] + nP \).

From the first transit, we can write the phase as \( t_c[0] = 4.91 \text{ days} \). The gap to the next transit is 7.33 days, so the orbital period should be some fraction of that, \( P = 7.33 \text{ days}/n[1] \). The gap between transits 2 and 3 is 11.02, so \( P = 11.02 \text{ days}/(n[2]-n[1]) \). Setting those two equal we get \( n[1]/n[2] = 2/5 \), so if \( n[1]=2 \) and \( n[2]=5 \), the period is \( P = 3.67 \text{ days} \). It’s also possible to be a factor of 2 — or another integer— smaller than that, but it’s a typical hot Jupiter with that period, so let’s suppose that’s right. The problem says we also have an RV orbit detected, so that could
firmly tell us the orbital period.

I like using $T$, which is $T_{\text{tot}}$, the interval between the halfway points of ingress and egress (sometimes referred to as contact times 1.5 and 3.5).” - Winn

It’s also the duration that the center of the planet is projected on the surface of the star.

I wrote a program to combine all three transits with the ephemeris equation and the trapezoidal model:
Normalized flux (using median of obviously out-of-transit points for each segment) vs. time (days)

My trapezoidal model is parameterized by:
P, $t_c[0]$, $\delta$, T, and slope,
Where slope is what I chose in place of $\tau$ from the chapter. ($\tau = \delta / \text{slope}$)

$P = 3.67353 \pm 0.00010$ days
$t_c[0]=4.9129 \pm 0.0004$ days
$\delta = 0.01723 \pm 0.00016$
$T = 0.1021 \pm 0.0005$ days
Slope_ingress = $-1.46 \pm 0.13$ day$^{-1}$
Slope_egress = $1.32 \pm 0.11$ day$^{-1}$

The two slopes are consistent to within their error bars (I just wanted to check) and therefore can be combined. I sum their absolute values as:

$\text{Var}_i = (\text{standard deviation}_i)^2$ [variance]

$\text{Var of the sum} = (\text{Var}_\text{ingress} + \text{Var}_\text{egress}) = 0.029$

Standard deviation of the sum = 0.17

Standard deviation of the average = Standard deviation of the sum / 2 = 0.09

Weighted average of the Slope = sum (Slope_i / Var_i) / sum (1/Var_i) = 1.38
Slope = 1.38 +/- 0.09 day^{-1}

Using the equation
\[ \tau = \frac{\delta}{\text{slope}} \]
And using linear error propagation formula:
\[ \sigma_{\tau} = \sqrt{\left( \frac{\sigma_d}{\text{slope}} \right)^2 + \left( \frac{\sigma_{\text{slope}} \cdot \delta}{\text{slope}^2} \right)^2} \]
\[ \tau = 0.0125 +/- 0.0008 \text{ days} \]

Ok, so let’s start converting this to physical parameters.

Hot Jupiters (~3 day orbits) are usually on circular orbits, and we can’t tell that precisely from the transit data, though the RV data should be able to tell. We’re told to assume it’s circular, e=0.

One easy parameter (without limb darkening) is the ratio of radii. It is:
\[ k = \frac{R_p}{R_{\text{star}}} = \frac{\sqrt{\delta}}{0.1313 +/- 0.0006} \]
Where standard deviation comes from the error propagation formula:
\[ \text{Sigma}_\delta \cdot \text{abs}\left( \frac{d\delta}{dk} \right) = 0.00016 \cdot \text{abs}\left( \frac{1}{2\sqrt{\delta}} \right) \]

If the orbit were edge-on, then the ratio if ingress time \( \tau \) to transit duration \( T \) would be almost exactly equal to the ratio of radii, \( k \). It is: 0.122(8), which is consistent with \( k \). Actually, we expect the transit duration to go as \( \sqrt{1-b^2} \), where \( b \) is the normalized impact parameter (Winn equation 18). Also, in the low-b limit, the ingress time should go as \( 1/\sqrt{1-b^2} \), because it takes longer to cross the edge of the star if it’s encountering it at an angle. So the ratio should go as \( 1/(1-b^2) \). The upper limit on the ratio is 0.146 (3-sigma), which is a factor of 1.11 larger than \( k \), for an upper limit of \( b=0.31 \).

So, assuming an edge-on orbit makes it easy to convert the \( K \) amplitude to a mass, as \( \sin i = 1 \).
Equation 14 from Lovis & Fischer gives:
\[ K_1 = 141.24 \text{ m/s} = 28.4329 \text{ m/s} \cdot \frac{m_p}{M_{\text{Jup}}} \cdot \left( \frac{(m_p+M_{\text{star}})/M_{\text{sun}}}{P/1\text{yr}} \right)^{2/3} \]
Which I solved iteratively by first guessing \( m_p=0 \), then plugging the result into the parenthesis to get:
\[ m_p = 1.00024(1) M_{\text{Jup}} \] -- where the error is in parenthesis wrt the final digit, obtained by propagating the fractional error on \( P \) (i.e., times \( 1/\pi \)) to the final value.

Now we can get the semi-major axis, combining the masses with the orbital period:
\[ a = \frac{G (M_{\text{star}}+m_p)^{1/3}}{P/2\pi} \]
This gives:
\[ a = 0.0450035(8) \text{ AU} \]

Finally, we can use the duration, period, and semimajor axis to get the stellar radius according to Winn’s equation 19:
\[ R_{\text{star}} = \pi a T_0 / P. \]

Our b is consistent with 0, so taking \( T_0 = T \), we have \( R_{\text{star}} = 0.845(4) R_{\text{Sun}} \)

To summarize, we have the mass and radius of both bodies, the semi-major axis in physical units, the orbital orientation (edge-on).

We do not know how the system is tilted on the sky relative to local north -- both photometric transits and radial velocity measurements are invariant to rotations around the line of sight.

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(2) Comparison of planet detection / characterization methods

(I) An equation that gives the amplitude of the signal that reveals the planet's existence.

(II)

(a) Radial velocity.

(I) Lovis & Fischer give (eqn. 13):
\[ K = 28.4329 \text{ m/s} \ m_p \sin i / M_{\text{Jup}} \left((m_p + M_{\text{star}})/M_{\text{sun}}\right)^{1/2} \left(a/1\text{AU}\right)^{-1/2} / \sqrt{1-e^2} \]

(II) For the purpose of simplifying this to a rough detectability boundary, we'll take \( m_p << M_{\text{star}}, e=0, \sin i = 1 \), and do a survey of Sun-mass stars, leaving:
\[ K = 28.4329 \text{ m/s} \ m_p / M_{\text{Jup}} \left(a/1\text{AU}\right)^{-1/2} \]

Let's do a survey for 10 years with 2 m/s precision per point, and take 100 points per star. Then the noise will beat down to \( 2/\sqrt{100} = 0.2 \) m/s, but we will require \( \sim 5 \) sigma detections because we can search for planets at lots of different periods (there's a look-elsewhere penalty to pay). Hence the limit will be \( K=1 \) m/s out to \( (10 \text{ AU})^{2/3} \rightarrow 4.6 \text{ AU} \). So less than that, we have:
\[ m_p = 0.035 M_{\text{Jup}} \left(a/1\text{AU}\right)^{1/2} \]

Beyond that, we can only measure accelerations, which fall off as \( 1/a^2 \). So we can connect the two there and get:
\[ m_p = 0.075 M_{\text{Jup}} \left(a/4.6\text{AU}\right)^2. \]

(b) Transit.

(I) For a single transit, the depth is \( \delta \sim (R_p/R_{\text{star}})^2 \).

(II) This is a bit more complicated. Suppose you have a telescope that can collect a hundred million photons every 10 minutes from a particular star. Then the 10-minute datapoint has a Poisson fluctuation of \( \sqrt{10^8}=10^4 \). That means a \( 10^{-4} \) signal could be detected at 1-sigma in 10 minutes. Let's scale things off the Earth, which has \( R_p/R_{\text{sun}} = 109 \). Then we need to detect an \( 8*10^{-5} \) signal, and say we need to detect it at 7-sigma confidence because there's lots of places transits could pop up (a big look-elsewhere penalty). The error on the 10-minute integrations will be beat down as the square root of the number of measurements. So we need
10^2 measurements in transit. That's 10^3 min \approx 17 hours in transit. Winn’s equation 19 has
\[ T = R_{\text{star}} P/(\pi a) = 13 \text{ hr} \left( a/1\text{AU} \right)^{1/2}. \]
By putting \( P = a^{3/2} \) and assuming a Sun-like star.
So at a distance of \( a=1.71 \text{ AU} \), or a period of 2.2 yr, this single transit would be detectable. (In
reality, in order to measure the orbital period, people have said we need 3 transits… but this is
the basics.)

So let's do a 2.2 year survey of Sun-like stars for which we can get 10^8 photons per 10
minutes. Then planets with periods \( P>>2.2 \text{ years} \) will be undetectable because they just don’t
transit while we’re looking, and periods \( P<<2.2 \text{ years} \) will transit many times \( n=(2.2\text{yr}/P) \). Then
their total time in transit will be:
\[ nT \sim 1/a \sim 17 \text{ hr} \left( a/1.71 \text{AU} \right)^{-1}. \]
Then the signal-to-noise will be a factor of \( (a/1.71\text{AU})^{1/2} \) bigger than we computed, so the limiting
signal \( \delta \) can be that factor smaller, and \( R_p \) will scale as \( a^{1/4} \). So finally, we get:
\[ R_p = R_{\text{Earth}} \left( a/1.71 \text{AU} \right)^{1/4}. \]

We can convert this limit into mass via a constant-density model for rocky planets (sub-Earths):
\( M_p/M_{\text{Earth}} = (R_p/R_{\text{Earth}})^3 \), to arrive at:
\[ M_p = M_{\text{Earth}} \left( a / 1.71 \text{AU} \right)^{3/4}. \]

In practice, people seem to always want to see 3 transits before declaring a planet detected,
particularly if it’s near the signal-to-noise limit. So in reality its sensitivity abruptly shuts off at
about 1 AU (for the 4 year Kepler survey).

(c) Direct Imaging

(I) The detectability is a function of the intrinsic luminosity. That “amplitude” can be given
in a power-law form based on slide 6 of lecture 6. That chart becomes (taking \( M_{\text{Jup}} = 0.001 M_{\text{Sun}} \)):
\[
\begin{array}{ccc}
\text{Log } M/M_{\text{Jup}} & \text{Log } L/L_{\text{sun}} & \text{Log age/Myr} \\
0 & -4.9 & 6 \\
0 & -8.1 & 9 \\
1 & -2.8 & 6 \\
1 & -6.2 & 9
\end{array}
\]
So we can put these together in a 2-variable power-law:
\[
\text{Log } L/L_{\text{sun}} = 1.75 + 2.00 \times \text{Log } M/M_{\text{Jup}} - 1.10 \times \text{Log age/Myr}
\]
This is still not exactly the amplitude of the signal, because we need to choose a band in which
to observe the planet. But it is good enough for a rough boundary of detectability.
As emphasized in lecture, the contrast with the star’s luminosity is the biggest challenge for
detecting planets, and it forms the inner boundary to which they can be detected. The angular
scale for imaging to get to these low levels is several \( \lambda/D \), where \( \lambda \) is the
wavelength of the observations and \( D \) is the diameter of the telescope. A 2 micron wavelength
using an 8 m telescope would have \( \lambda/D = 0.32 \text{ arcsec} \). So beyond 0.6 arcsec, say, forms
our inner boundary.

(II) Ok, here we would like to image nearby stars that are \( \approx 100 \text{ Myr} \), so that the planets are
bright enough, which is only \( \approx 1/50 \) of the Sun’s age. The closest star is about \( 1 \text{ pc} \) away, and
the occurrence of giant planets is around ~1/10 of stars. So one will need to look at the 500th closest star to find just one planet. The distance would then be \((500)^{1/3}\) 1 pc = 8 pc. For a real survey, you’d want to expect to detect ~10 planets, so go out to \((5000)^{1/3}\) 1 pc = 17 pc. Now, it's not really true that these young stars are evenly distributed in the Galaxy, as one orbit around the Galaxy takes 100 Myr, therefore stars that young still live pretty close to their birthplace -- in “moving groups” of stars that were formed together and now unbound and drifting apart.

The upshot is that at 17 pc, a 0.6 arcsec “inner working angle” (where the star becomes too bright to find a planet) maps to \(17\times0.6 = 10\) AU. The limit in mass is determined by how much time we can spend integrating; one must balance digging deeper on an individual star and covering enough stars to actually find a significant number of planets. The lowest mass planet found so far by direct imaging is \(~2\,M_{\text{Jup}}\) (51 Eri b). We’ll take that as the limit in mass.

\[(d)\text{ Microlensing}\]

\[(I)\text{ This is a bit tricky. The basic signal, of one star going behind another has an amplitude that's easy to express as an equation:}\]

\[A = \left(\frac{u^2 + 2}{u \sqrt{u^2 + 4}}\right),\]

where \(u\) is the projected distance in terms of the Einstein radius \(r_E\). The planetary signal is the degree to which the planet’s presence warps the shape away from the family of curves defined by rectilinear motion of \(u(t)\) (Gaudi’s equation 8). For that, it appears no succinct equation exists. If the magnification is high due to a close passage of the lens star in front of the source star, there is extreme sensitivity to finding the planet near the Einstein ring. For typical numbers of a source in the bulge of the Galaxy, and a low-mass lens in the foreground, this works out to \(r_E=2.2\) AU (Gaudi eq. 4). For planets not at the Einstein radius, the amplitude of their perturbation falls off with \(1/r_{Ep}\) where \(r_{Ep}=(m_p/m_{\text{star}})^{1/2} r_E\). So the width in semi-major axis space that is probed by this method is order unity far for equal-mass bodies and scales as the square root of \(m_p\) when \(m_p<<m_{\text{star}}\).

Also, the planet must perturb the Paczynski curve (Gaudi’s figure 3) far enough from the single-lens case so that it is noticed. As discussed in Gaudi’s section 4.2, the limiting mass depends on finite-source effects, and it can go down to Mars mass (0.1 Earth masses), so even right at the Einstein radius of a perfect alignment, that is the scale.

\[(II)\text{ Our survey would point to the Galactic bulge to get the highest number of microlensing possible in the field of view (biggest background density). Recall that the microlensing optical depth (the probability that a foreground star happens to be one Einstein ring in projection on top of a background star) is only 1 in a million. So many millions of stars need to be monitored in time. Suppose we monitor 100 million, and watch for 10 Einstein-ring-crossing times (several years). Then we can expect ~1000 stellar microlensing events -- how many of those will have a planetary signal? In order to probe the smallest planetary masses, we want to see down to main sequence stars (so that the finite source effect is minimized) -- so a large telescope could be called for. The need to resolve the stars means going into space would be much better; adaptive optics systems on the ground cannot correct a wide field of view.}\]

Using the probability of lensing above, we have:
\[ m_p = m_{\text{star}} \frac{(a-r_E)^2}{r_E^2} \]

With a floor at 0.1 \( m_{\text{Earth}} = 3 \times 10^{-4} m_{\text{Jup}} \).

Let’s say that of these 1000 events, the Einstein ring radius is peaked at 2.2 AU, but has a standard deviation in the \( \log a \) of 0.5. So then of the 1000 stellar microlensing events, the reason you probably won’t catch a low-mass planet at 0.2 AU is because it’s rare for such a planet to fall within \( da=a-r_E \) of the few Einstein ring that are this small, because they are so rare there. Since the width of a \( \frac{1}{3} M_{\text{Earth}} \) planet is \( \sim 1/10^3 \), that’s the size that’s likely to get seen, just at 2.2 AU. Then 30 \( M_{\text{Earth}} \) planets will be seen out to Einstein ring radii that are \( \sim 1/10 \) less dense, i.e. 1.5 sigma, or from 1.45 AU to 2.95 AU. A 10 \( M_{\text{Jup}} \) planet could be seen out to a place where Einstein rings are \( 1/100 \) less dense, i.e. 2.15 sigma, or from 1.125 AU to 3.275 AU. So I will use these sets of points on my plot.

(e) Timing

(I) The amplitude is \( m_p \sin i / m_{\text{star}} (a / c) \), up to the survey length, and then just like the radial velocity technique, you can see the acceleration \( \text{accel} = G m_p \sin i / a^2 \) beyond the survey length of \( T \), amounting to an timing displacement (signal) of \( \frac{1}{2} \text{accel} (T/2)^2 / c \) (O-C would be a parabola, centered in the middle of the dataset and curving up or down, \( T/2 \) to each side.)

\[ = 2 G m_p \sin i T^2 / (a^2 c) \]

(II) We can easily implement a timing survey using the \( T=4 \)-year baseline of Kepler data, using eclipse times. The precision on number of datapoints varies (due to different photon noise, depths, and orbital periods of different binaries), but notionally, let’s assume a 30 second error bar on each of 100 eclipses over the four years. Then the amplitude we can probe is \( 30 \times \text{sqrt(100)} = 3 \) seconds. But we want to detect it at \( \sim 5 \) sigma, due to having to search for the period in the data, so say 15s amplitude is detectable. Assume the hosts are two Sun-like stars, generally, so \( m_{\text{star}}=2m_{\text{sun}} \). And \( \sin i \) is statistically often near 1. The survey length can see an entire orbit out to

\[ a_{\text{survey}} = (Gm_{\text{star}})^{1/3} (T/2\pi)^{2/3} = 3.17 \text{ AU} \]

Then we have

\[ m_{p,\text{lim}} = 20 M_{\text{Jup}} \left( a / 3.17\text{AU} \right) \] for \( T<4 \) years, \( a<3.17\text{AU} \).

And beyond that, we have:

\[ m_{p,\text{lim}} = 4 M_{\text{Jup}} \left( a / 3.17\text{AU} \right)^2 \] for \( T>4 \) years, \( a<3.17\text{AU} \).

(f) Astrometry

(I) The measurement is an angular amplitude, amounting to \( m_p / m_{\text{star}} (a / d) \), where \( d \) is the distance to the system. The sensitivity would likely be less at orbital frequencies within \( \sim 1/T \) of 1 year\(^{-1} \), where \( T \) is the survey length (because the signal of the parallax would not be strictly separated from the signal of the planet, at a frequency that close).

(II) GAIA survey should go about 5 years, and have a precision of about \( 10^{-4} \) arcsec:

http://www.cosmos.esa.int/web/gaia/science-performance
Let’s consider 100 measurements, and require 5-sigma detections, therefore the limiting amplitude is $5 \times 10^{-5}$.

For the target stars, we could do a similar calculation to the survey contemplated for Direct imaging, except that the stars are not required to be young. The issue here is that the brightest stars saturate, so the very closest stars cannot be measured. So let’s consider the main sample to be at 100 pc, consisting of Sun-like stars. We could use $1\text{ arcsecond} = (1/206265)\text{ radian}$, and $1\text{ pc} = 206265\text{ AU}$, and directly evaluate the mass/semi-major axis limit. Or we could convert the limiting precision to AU at the 100 pc distance: $5 \times 10^{-3}\text{AU}$. Then the largest closed semi-major axis in the 5-year survey is $5^{2/3} = 2.92\text{ AU}$, so the limiting mass is:

$$m_p = m_{\text{star}} \times \left(5 \times 10^{-3}\text{AU}/2.92\text{AU}\right) \left(a/2.92\text{AU}\right)^{-3/2}$$

$$= 1.8 m_{\text{Jup}} \left(a/2.92\text{AU}\right)^{-1} \quad \text{for T<5 years, a<2.92 AU}$$

Similarly to the timing signal, on orbital periods longer than the baseline, we will get an acceleration whose limiting mass will be ~5 times lower at this cut-off, then scale as $a^2$. So

$$m_p = 0.4 m_{\text{Jup}} \left(a/2.92\text{AU}\right)^2 \quad \text{for T>5 years, a>2.92 AU}$$

Plot with all these constraints….

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Synergies:

The obvious synergies are between radial velocity and transit, as in the last problem, we can get both the planet mass and radius, allowing for investigations into planetary structure.
Direct imaging plus astrometry would give you planetary luminosity and mass. Combining that with planetary models could give the age of the (cooling) planet. It’s most likely for close systems; maybe it will be possible for beta Pic b.