Astronomy 313000
Spring Quarter 2007

Problem Set 3

Due: Friday, May 18

If you have any questions, please contact me *before* the due date by e-mail: frieman@fnal.gov.

1. The dark halo density profile predicted by cold dark matter models of structure formation can be reasonably well fit by the 'universal' function of Navarro, Frenk, and White (1996, NFW):

$$\rho(r) = \frac{\rho_s}{(r/r_s)[1 + (r/r_s)]^2}$$

where ρ_s and r_s are the characteristic density and radius. Assuming a spherical halo, find an analytic expression for the rotation speed v_c as a function of r; here, 'analytic' means solve it yourself in terms of elementary functions rather than simply feeding it to Mathematica. Plot $v_c/(G\rho_s r_s^2)^{1/2}$ as a function of r/r_s (Mathematica is acceptable for plotting). How does v_c scale with r at the smallest scales, $r \ll r_s$?

2. The pseudo-isothermal density profile with core radius r_c has the form

$$\rho(r) = \frac{\rho_0}{1 + (r/r_c)^2}$$

.

Unlike the NFW or singular isothermal sphere profiles, the pseudo-isothermal model has finite central density. Find an analytic expression (see above for definition of analytic) for the rotation speed as a function of r and plot $v_c/(G\rho_0 r_c^2)^{1/2}$ as a function of r/r_c. Compare qualitatively with the result of the previous problem; how does v_c depend on r at $r \ll r_c$?

3. Consider a self-gravitating spherical distribution of stars, $\rho(r)$. Suppose that at small radii from the center of the stellar distribution, the stars are isothermal, i.e., the velocity dispersion $\sigma^2 = P/\rho =$ constant. Suppose further that the stellar density profile is given by

$$\rho(r) = \frac{\rho_0}{[1 + (r/a)^\alpha]^\beta}$$

where a is the core radius.

   (a) Assuming the stellar system is in hydrostatic equilibrium, by considering the behavior at $r \ll a$ determine the value of the exponent α.

(b) Suppose we also require that $\rho \propto r^{-3}$ at very large radius; determine the exponent β.

(c) What is the relation between σ, a, and ρ_0?

(d) Find an analytic expression for the mass enclosed within radius r.

(e) Plot the circular speed, in units of $(G\rho_0 a^2)^{1/2}$, vs. r/a. At what value of r/a does the circular speed peak? How does v_c depend on r at large radii $r \gg a$?

(f) Assume a constant stellar mass to light ratio, $Y = \rho(r)/j(r)$, where $j(r)$ is the 3-d stellar luminosity density. Find an analytic expression for the projected surface brightness profile $I(R)$ in terms of R, a, ρ_0, and Y.

4. We have assumed that spiral rotation curves extending to large galactocentric radii imply the existence of dark halos, since the inferred mass to light ratios are very large compared to those for stellar populations. Milgrom and Bekenstein proposed an alternative hypothesis to account for flat rotation curves, known as modified Newtonian dynamics (MOND). In MOND (in its purest form at least), there is no dark matter—instead Newton's 2nd law is modified at very small acceleration a, to $F = maf(a/a_0)$, where $f \to 1$ for $a \gg a_0$ recovers ordinary Newtonian dynamics at the large accelerations probed in the laboratory (and in everyday life).

(a) To obtain a flat rotation curve at radii large compared to the effective optical radius for a spiral disk, what form should the dimensionless function f take at small a/a_0? Use this simple asymptotic form for f for the rest of this problem.

(b) Since there is no dark matter in MOND, we can in principle infer galaxy masses from their stellar luminosities and knowledge of their stellar populations (and gas content). Assuming all spirals have similar stellar+gas mass to light ratios, what does MOND predict for the form of the scaling relation between asymptotic galaxy rotation speed and galaxy luminosity? Work in the same $a \lesssim a_0$ limit as above.

(c) Suppose the Tully-Fisher relation between luminosity and asymptotic circular speed for spirals can be approximated (in some band) by

$$\frac{L}{2 \times 10^{10} L_\odot} = \left(\frac{v_c}{200\mathrm{km/sec}} \right)^4$$

and that the stellar+gas mass to light ratio is $M/L = 3M_\odot/L_\odot$ for all spirals in the same band. Estimate the numerical value of the parameter a_0 in cgs units (i.e., cm sec^{-2}), again assuming that asymptotic rotation speeds are in the limit $a \lesssim a_0$. Is that assumption self-consistent?

(d) Why is this model less popular than dark matter?

5. Galaxy lenses are often modeled using the singular isothermal sphere (SIS) model, for which the density profile is given by

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

(a) Write down the expression for the Einstein angle θ_E in terms of the velocity dispersion σ_v, and the distances D_{LS}, D_S, and D_L.

(b) Consider a light ray that passes the lens at impact parameter b from its center, where b is measured perpendicular to the line joining the lens center to the observer. Write down the expression for the convergence, i.e., the dimensionless surface mass density $\kappa(b) = \Sigma(b)/\Sigma_{crit}$, in terms of θ_E and b.

(c) Derive an expression for the tangential shear $\gamma_T(b)$ in terms of the same quantities.

(d) Solve the lens equation and thus derive an expression for the amplification of each of the images in terms of the angular position θ of the image (relative to the line joining lens and observer) and the Einstein angle θ_E. Sketch or plot curves for the amplification of each image as a function of β/θ_E, where β is the angular position of the source relative to the line of sight from observer to lens if the light is not bent.

6. This problem concerns microlensing by compact objects (MACHOs) in the halo of our galaxy. Assume for simplicity that the observer and source stars (whose images are magnified by lensing) are at rest.

(a) Calculate the angular speed μ in milliarcsec per year of a lens at distance $D_L = 10$ kpc and with speed transverse to the line of sight of $v = 200$ km/sec. How does your answer scale with v and D_L?

(b) For $D_L = 10$ kpc and $D_S = 50$ kpc (the distance to the LMC), estimate the Einstein angular radius θ_E in milliarsec for a MACHO of 1 solar mass; how does the result scale with MACHO mass?

(c) If the angular distance of closest approach of the lens and source is exactly equal to θ_E, what is the maximum amplification factor of the source (including both images, since they are unresolved)?

(d) For a lens at distance of 10 kpc, source at distance of 50 kpc, MACHO of 1 solar mass, and transverse velocity as above, calculate the characteristic duration of a lensing event, $\tau = \theta_E/\mu$. How does the result scale with MACHO mass, D_L, D_L/D_S, and v?

(e) Consider the following idealized geometry and model. Assume the Sun is 8 kpc from the Galactic Center. Imagine that the LMC stars (which are being lensed) are precisely 50 kpc from us and that they are (all) positioned on the extension of the radial line from the Galactic Center to the Sun (and on the same side of the Galactic Center as the Sun; this is clearly oversimplified: the LMC does *not* lie in the plane of the Milky Way disk). Model the Milky Way halo as a spherical SIS, with mass density profile $\rho_h(r) = v_c^2/4\pi G r^2$, with a circular speed that matches the asymptotic rotation curve of our galaxy, i.e., $v_c = 220$ km/sec. If the halo is composed entirely of MACHOs, estimate the microlensing optical depth to the LMC, i.e., the probability that a star in the LMC is being lensed at any given time. For this estimate, define an LMC star to be lensed if its angular separation on the sky from a MACHO is θ_E or less, where θ_E is the Einstein radius (which depends on the distance to the lens). Thus, roughly how many LMC stars should be monitored to obtain a reasonable number of lensing events if MACHOs make up the entire halo?

(f) The MACHO project monitored the brightnesses of stars in the LMC for a number of years, using 300 sec exposures on a 1.3m telescope. This translated roughly into a magnitude limit (in V band) of $m_V = 20$. In the standard MK stellar spectral sequence (OBAF...), which main sequence stellar types in the LMC could they detect and which not? Could they detect most of the main sequence stars in the LMC? Could they detect most of the giant stars? For these estimates, ignore the effects of reddening, extinction, and metallicity.