

Due Friday, June 4

Astronomy 304 Spring Quarter 2010 : Problem Set 3

1.) If the Riemann curvature is isotropic, then the curvature tensor has the form

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})$$

where  $K$  is a scalar.

(a.) Derive an expression for the Ricci curvature  $R_{\alpha\beta}$  in terms of  $K$  and the metric  $g_{\alpha\beta}$ .

(b.) Derive an expression for the Ricci scalar  $R$ :

(c.) Thus derive an expression for the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R.$$

(d.) Show that the Bianchi identity, which can be expressed as  $\nabla^\alpha G_{\alpha\beta} = 0$ , where  $\nabla^\alpha$  is the covariant derivative, implies that  $K$  must be a constant. Thus isotropy at every point implies homogeneity.

2.) Consider the spatial part of the FRW metric as the metric for a 3-dimensional world at a given time:

$$\begin{aligned} ds^2_{(3)} &= h_{ij} dx^i dx^j = \cancel{\dots} \\ &= a^2 \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad k = \pm 1, 0 \end{aligned}$$

where the scale factor  $a$  is a constant (for a constant-time slice). This metric has isotropic curvature, so its Riemann tensor has the form:

$${}^{(3)}R_{ijkl} = \frac{k}{a^2} [h_{ik}h_{jl} - h_{il}h_{kj}]$$

- (a.) In terms of  $h_{ij}$ ,  $k$ , and  $a$ , use this relation to find expressions for  ${}^{(3)}R_{ij}$  and  ${}^{(3)}R$ , the 3-dimensional Ricci curvature and Ricci scalar. Give an explicit expression for  ${}^{(3)}R_{\theta\theta}$ .
- (b.) Verify the result above for  ${}^{(3)}R_{\theta\theta}$  by calculating it directly from the affine connection ( $P^i_{jk}$ ) for the metric  $h_{ij}$  above.

3.) Consider the first-order Friedmann equation for a matter-dominated Universe with no dark energy ( $\Lambda=0$ ). By converting to conformal time,  $d\eta = dt/a(t)$ , where  $a(t)$  is the FRW scale factor, show that the solutions can be written as:

$$a(\eta) = \begin{cases} \frac{q_0(1-\cos\eta)}{H_0(2q_0-1)^{3/2}} & \text{for } k=+1 \\ \frac{1}{4} H_0^2 a_0^3 \eta^2 & \text{for } k=0 \\ \frac{q_0(\cosh\eta-1)}{H_0(1-2q_0)^{3/2}} & \text{for } k=-1 \end{cases}$$

Now invert the coordinate transformation above, i.e., integrate  $\eta$  to show that the age of the Universe can be written:

$$t = \begin{cases} \frac{q_0}{H_0(2q_0-1)^{3/2}} (\eta - \sin\eta) & \text{for } k=+1 \\ \frac{1}{12} H_0^2 a_0^3 \eta^3 & \text{for } k=0 \\ \frac{q_0}{H_0(1-2q_0)^{3/2}} (\sinh\eta - \eta) & \text{for } k=-1 \end{cases}$$

4.) a.) Assuming what you know about the present photon and neutrino temperature, and assuming 3 species of massless neutrinos, find the redshift and temperature at the epoch of matter-radiation equality in terms of the matter density  $\Omega_m$  and  $h$ . Express  $k_B T_{eq}$  in electron volts (eV). Evaluate  $k_B T_{eq}$  for  $\Omega_m = 0.25$  and  $h = 0.7$ .

b.) Assuming  $\Omega_b = 0.04$  and  $h = 0.7$ , use the Saha equation to determine the temperature  $T_{rec}$  at which hydrogen recombines. For this problem, define recombination to be the point when the equilibrium fractional ionization drops to  $X_e = 0.5$ . Again express the answer for  $k_B T_{rec}$  in eV.

c.) To what do you attribute the near-coincidence between  $T_{rec}$  and  $T_{eq}$ ? Comment.

5.) This problem examines the dependence of the  ${}^4\text{He}$  mass fraction from Big Bang Nucleosynthesis on the number of light neutrino species. Although we knew of  $N_\nu = 3$  neutrino species ( $\nu_e, \nu_\mu, \nu_\tau$ ) , people have speculated about a possible 4th generation. An extra species of light neutrino would increase the number of effective relativistic degrees of freedom at the time of BBN ,  $g_*(T)$ .

In class, we estimated the  ${}^4\text{He}$  yield as

$$X_4 = 2X_n(t_{\text{BBN}}) = 2X_n(t_F) \exp(-t_{\text{BBN}}/t_F)$$

where  $t_{\text{BBN}} = 180$  sec is the time at which nucleosynthesis occurs (the D bottleneck breaks),  $t_F$  is the time of neutron-proton freeze-out, and  $T_n = 909$  sec is the neutron mean lifetime. Here,  $X_n$  is the neutron mass fraction,

$$X_n = \frac{(n_n/n_p)}{1 + (n_n/n_p)} \quad \text{with} \quad \left(\frac{n_n}{n_p}\right)_F = \exp\left(-(\bar{m}_n - \bar{m}_p)/T_F\right) \\ = \exp\left(-Q/T_F\right)$$

a.) Suppose we repeat the BBN calculation, but now with more neutrino species, i.e., with  $g_*(T)$  increased by an amount  $\delta g_*$  around the time of BBN. Derive an expression for the fractional change in the  ${}^4\text{He}$  yield,  $\delta X_4 / X_4$ , in terms of  $\delta g_* / g_*$ ,  $(n_\nu / n_p)_{T_F}$ ,  $(Q / T_F)$ , and  $t_{\text{BBN}} / \tau_n$ , where the latter quantities are to be evaluated for the 'fiducial' model with  $N_\nu = 3$  (i.e., with  $g_*(T_{\text{BBN}})$  equal to its standard value). Evaluate  $\frac{\delta X_4 / X_4}{\delta g_* / g_*}$  by using the fiducial values  $T_F = 0.8 \text{ MeV}$ ,  $t_{\text{BBN}} = 180 \text{ sec}$ , and  $\tau_n = 909 \text{ sec}$ .

b.) Relate  $\delta g_*$  to the excess number of light neutrino species, i.e. to  $\delta N_\nu = N_\nu - 3$ , and evaluate this relation. Finally, assuming a fiducial value  $X_4 = 0.235$ , find the numerical relation between  $\delta X_4$  and  $(N_\nu - 3)$ . This estimate is in quite good agreement w/ what comes out of the BBN codes.