

Thus,

$$\mathcal{D}_\beta \left(R_{\alpha\gamma}^\beta - \frac{1}{2} g_{\alpha\gamma}^\beta R \right) = 0$$

or

$$\mathcal{D}_\beta (G_{\alpha\gamma}^\beta) = 0$$

This is equivalent to

$$\mathcal{D}^\beta G_{\beta\gamma} = \mathcal{D}^\beta \left(R_{\alpha\gamma} - \frac{1}{2} g_{\alpha\gamma} R \right) = 0$$

These are the contracted Bianchi identities

2) The easiest way to proceed is to start by contracting:

$$R_{\alpha\beta\gamma\delta} = K (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

Contract on α, γ :

$$\begin{aligned} g^{\alpha\gamma} R_{\alpha\beta\gamma\delta} &= R_{\beta\delta} = K (g^{\alpha\gamma} g_{\alpha\delta} g_{\beta\gamma} - g^{\alpha\gamma} g_{\alpha\gamma} g_{\beta\delta}) \\ &= K (4g_{\beta\delta} - \delta_\delta^\alpha g_{\beta\alpha}) = K (4g_{\beta\delta} - g_{\beta\delta}) \end{aligned}$$

$$\therefore \underline{R_{\beta\delta} = 3K g_{\beta\delta}}$$

Further contraction gives

$$R = g^{\beta\delta} R_{\beta\delta} = 3K g^{\beta\delta} g_{\beta\delta} = 12K$$

Thus, the Einstein tensor is

$$G_{\beta\delta} = R_{\beta\delta} - \frac{1}{2} g_{\beta\delta} R = 3K g_{\beta\delta} - \frac{1}{2} g_{\beta\delta} (12K) = -3K g_{\beta\delta}$$

The contracted Bianchi identity, ~~also followed~~, is

$$\nabla^\beta G_{\beta\delta} = 0. \text{ Here, this gives}$$

$$\begin{aligned} 0 &= -3 \nabla^\beta (K g_{\beta\delta}) = -3 g_{\beta\delta} \nabla^\beta K = -3 g_{\beta\delta} \partial^\beta K \\ &= -3 \partial_\delta K \end{aligned}$$

where we've used the fact that the metric is covariantly conserved, $\nabla^\beta g_{\beta\delta} = 0$, and that K is a scalar, so $\nabla^\beta K = \partial^\beta K$. Thus, K must be a constant.

2. (a) Start with Riemann 3-curvature:

$${}^3R_{ijk}{}^l = \frac{k}{a^2} (h_{ik} h_{jl} - h_{il} h_{kj})$$

Contract on i, k to get Ricci tensor:

$$\begin{aligned} {}^3R_{jl} &= h^{ik} {}^3R_{ijk}{}^l = \frac{k}{a^2} (h^{ik} h_{ik} h_{jl} - h^{ik} h_{il} h_{kj}) \\ &= \frac{k}{a^2} (3h_{jl} - \delta^k_l h_{kj}) = \frac{k}{a^2} (3h_{jl} - h_{lj}) \end{aligned}$$

$$\therefore {}^3R_{jl} = \frac{2k}{a^2} h_{jl}$$

(here used $h^{ij} h_{ij} = 3$ in 3 dimensions)

\therefore Ricci 3-scalar is

$${}^3R = h^{jl} {}^3R_{jl} = \frac{6k}{a^2}$$

$$\text{Thus } {}^3R_{\theta\theta} = \frac{2k}{a^2} h_{\theta\theta} = \frac{2k}{a^2} (a^2 r^2) = 2kr^2$$

Now calculate directly: from the definition in class, we have

$${}^3R_{ij} = \partial_k T_{ij}^k - \partial_j \left(\frac{1}{\sqrt{h}} \partial_i \sqrt{h} \right) + T_{ij}^k \left(\frac{1}{\sqrt{h}} \partial_k \sqrt{h} \right)$$

$$- T_{in}^k T_{jk}^n$$

where the metric determinant

$$\sqrt{h} = (\det h_{ij})^{1/2} = \frac{a^3 r^2 \sin \theta}{(1 - kr^2)^{1/2}}$$

Thus,

$${}^3R_{\theta\theta} = \partial_r T_{\theta\theta}^r - \partial_\theta \left(\frac{1}{\sin \theta} \partial_\theta \sin \theta \right) + T_{\theta\theta}^r \left[\frac{(1 - kr^2)^{1/2}}{r^2} \partial_r \left(\frac{r^2}{(1 - kr^2)^{1/2}} \right) \right]$$

$$- T_{\theta\varphi}^\varphi T_{\theta\varphi}^\varphi - 2 T_{\theta r}^\theta T_{\theta\theta}^r \quad (\text{all other terms vanish})$$

The affine connections for this metric are:

$$T_{\theta\theta}^r = -r(1 - kr^2); \quad T_{\theta\varphi}^\varphi = \frac{\cos \theta}{\sin \theta}; \quad T_{\theta r}^\theta = \frac{1}{r}$$

Substituting these in above gives

$${}^3R_{\theta\theta} = -\partial_r \left[r(1 - kr^2) \right] + \csc^2 \theta - \frac{(1 - kr^2)^{3/2}}{r} \partial_r \left(\frac{r^2}{(1 - kr^2)^{1/2}} \right)$$

$$- \frac{1}{\tan^2 \theta} + 2(1 - kr^2)$$

$$= \frac{1}{2r} \partial_r(kr^4) = 2kr^2, \text{ which agrees with the expression above.}$$

3.) Starting with the usual first-order Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}p - \frac{k}{a^2}, \text{ change to conformal time:}$$

$$dy = \frac{dt}{a}, \text{ and assume matter-dominated: } p = p_0 \left(\frac{a_0}{a}\right)^3.$$

$$\text{Then } \frac{da}{dt} = \frac{da}{dy} \frac{dy}{dt} = \frac{1}{a} \frac{da}{dy}, \text{ so}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{1}{a^2} \frac{da}{dy}\right)^2 = \frac{8\pi G}{3} p_0 \left(\frac{a_0}{a}\right)^3 - \frac{k}{a^2}$$

Multiply by a^3 and take square-root:

$$\begin{aligned} a^{1/2} \frac{da}{dy} &= \left[\frac{8\pi G}{3} p_0 \left(\frac{a^3}{a_0}\right) - k a \right]^{1/2} \\ &= 2 \frac{d}{dy} (a^{1/2}) \end{aligned}$$

Invert this to solve for $\eta(a)$:

$$\frac{1}{2} dy = \frac{1}{2} \eta = \int_0^{a^{1/2}} \frac{d(a^{1/2})}{\left[\frac{8\pi G}{3} p_0 \left(\frac{a^3}{a_0}\right) - k a\right]^{1/2}} \equiv I(a, a_0, k)$$

Let $x = a^{1/2}$, ~~and~~ and define $c^2 = \frac{8\pi G}{3} p_0 a_0^3$ constant.

Then, for $k=+1$: $I(a, a_0, +) = \int_0^{a^{1/2}} \frac{dx}{[c^2 - x^2]^{1/2}} = \sin^{-1}\left(\frac{x}{c}\right) \Big|_0^{a^{1/2}} = \sin^{-1}\left(\frac{a^{1/2}}{\left(\frac{8\pi G}{3} p_0 a_0^3\right)^{1/2}}\right)$

$k=0$: $I(a, a_0, 0) = \int_0^{a^{1/2}} \frac{dx}{c} = \frac{a^{1/2}}{c} = \frac{a^{1/2}}{\left(\frac{8\pi G}{3} p_0 a_0^3\right)^{1/2}}$

$$k=-1 : I(a, a_0, -1) = \int_0^{H_2} \frac{dx}{(c^2 + k^2)^{1/2}} = \sinh^{-1}\left(\frac{x}{c}\right) \Big|_0^{H_2}$$

$$= \sinh^{-1}\left[\frac{a^{1/2}}{\left(\frac{8\pi G}{3} p_0 a_0^3\right)^{1/2}}\right]$$

Now invert these to give $a(n)$:

$$a(n) = \begin{cases} \frac{8\pi G}{3} p_0 a_0^3 \sin^2\left(\frac{n}{2}\right), & k=+1 \\ \frac{8\pi G}{3} p_0 a_0^3 \frac{n^2}{4}, & k=0 \\ \frac{8\pi G}{3} p_0 a_0^3 \sinh^2\left(\frac{n}{2}\right), & k=-1 \end{cases}$$

We want to express this in terms of H_0, q_0 :

From the 2nd order FLW equation, $\frac{\ddot{a}}{a} = -4\pi G p$, we

have $q_0 = -\frac{1}{H_0^2} \frac{\ddot{a}}{a} \Big|_{a_0} = \frac{4\pi G}{3} \frac{p_0}{H_0^2}$

The first-order FLW equation, evaluated at present epoch, gives:

$$H_0^2 = \frac{8\pi G}{3} p_0 - \frac{k}{a_0^2} \Rightarrow 1 = \frac{8\pi G}{3 H_0^2} p_0 - \frac{k}{a_0^2 H_0^2}$$

Thus, $1 = 2q_0 - \frac{k}{H_0^2 a_0^2}$ or $a_0^2 = \frac{k}{H_0^2 (2q_0 - 1)}$

for $k = \pm 1$.

$$\text{Thus, } \frac{8\pi G}{3} p_0 a_0^3 = 2q_0 H_0^2 a_0^3 = 2q_0 H_0^2 \left[\frac{k}{H_0^2 (2q_0 - 1)} \right]^{3/2} = \frac{2q_0}{H_0} \left(\frac{k}{2q_0 - 1} \right)^{3/2} \quad (k=\pm 1)$$

For $k=0$, we have $H_0^2 = \frac{8\pi G}{3} p_0$, so $\frac{8\pi G}{3} p_0 a_0^3 = H_0^2 a_0^3$

Substituting these in above gives finally:

$$a(n) = \begin{cases} \frac{2q_0}{H_0(2q_0-1)^{3/2}} \sin^2\left(\frac{n}{2}\right) & k=+1 \\ \frac{1}{4} H_0^2 a_0^3 n^2 & k=0 \\ \frac{2q_0}{H_0(1-2q_0)^{3/2}} \sinh^2\left(\frac{n}{2}\right) & k=-1 \end{cases}$$

Using the trig. identities: $\cos 2\eta = 2\cos^2\eta - 1$, have

$$\sin^2\left(\frac{n}{2}\right) = 1 - \cos^2\left(\frac{n}{2}\right) = \frac{1}{2}(1 - \cos n), \text{ and}$$

$$\sinh^2\left(\frac{n}{2}\right) = \cosh^2\left(\frac{n}{2}\right) - 1 = \cancel{\cosh^2\left(\frac{n}{2}\right)} \frac{1}{2}(\cosh n - 1)$$

and we arrive at the form stated in the problem.

To find the age t , invert the transformation

$$\text{above: } dt = a(n) dn \Rightarrow t = \int a(n) dn :$$

$$k=+1: t = \frac{q_0}{H_0(2q_0-1)^{3/2}} \int (1 - \cos n) dn = \frac{q_0}{H_0(2q_0-1)^{3/2}} (n - \sin n)$$

$$k=0: t = \frac{1}{4} H_0^2 a_0^3 \int n^2 dn = \frac{1}{12} H_0^2 a_0^3 n^3$$

$$k=-1: t = \frac{q_0}{H_0(1-2q_0)^{3/2}} \int (\cosh n - 1) dn = \frac{q_0}{H_0(1-2q_0)^{3/2}} (\sinh n - n)$$

~~Problems Set 3: Solutions~~

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4.) a.) Find z_{eq} and T_{eq} , assuming photons + 3 massless neutrinos. Assuming the present CMB temp. is $T_0 = 2.73 \text{ K}$, the present radiation density is

$$\rho_r^0 = \frac{\pi^2}{30} g_r T_{0,r}^4 = \frac{\pi^2}{30} (3.36) (2.73)^4 = 7.9 \times 10^{-34} \frac{\text{gm}}{\text{cm}^3}$$

where $g_r^0 = 3.36$ for 3 massless v's with $T_v^0 = \left(\frac{4}{11}\right)^{1/3} T_{0,r}$

Thus $\Omega_m h^2 = \rho_r^0 / (\hbar^2 \rho_{crit}) = 4.2 \times 10^{-5}$, using $\rho_{crit} = 1.88 \times 10^{-29} \text{ gm/cm}^3$. Since $\rho_r \sim a^{-4}$ and $\rho_m \sim a^{-3}$, we have

$$1 + z_{eq} = \frac{a_0}{a_{eq}} = \frac{\rho_m^0}{\rho_r^0} = \frac{\Omega_m h^2}{\Omega_r h^2} = \frac{\Omega_m h^2}{4.2 \times 10^{-5}} = \boxed{2.4 \times 10^4 \Omega_m h^2}$$

Since $T \sim a^{-1}$, we have

$$T_{eq} = T_0 (1 + z_{eq}) = (2.73 \text{ K}) (2.4 \times 10^4 \Omega_m h^2) \left(\frac{1 \text{ eV}}{1.16 \times 10^4 \text{ K}} \right)$$

$$\therefore \boxed{k T_{eq} = 5.6 \Omega_m h^2 \text{ eV}}$$

For $\Omega_m = 0.3$, $h = 0.65$, this gives

$$\boxed{k T_{eq} = 0.7 \text{ eV}}$$

b.) The equilibrium ionization factor is determined by the Saha equation,

$$\frac{1 - X_e}{X_e^2} = \frac{4\sqrt{2}}{\pi} S(3) \eta \left(\frac{kT}{m_e c^2} \right)^{3/2} \exp\left(\frac{B}{kT}\right)$$

where $\eta = 2.68 \times 10^{-8} D_b h^2$, $T = 2.73 K(1+z)$, $B = 13.6 \text{ eV}$, ~~and~~ $S(3) = 1.202\dots$, and $m_e c^2 = 0.511 \text{ MeV}$

for $D_b h^2 = 0.02$, one finds $X_e = 0.5$ at $\underline{kT_{\text{rec}}} = 0.32 \text{ eV}$

c.) T_{rec} is determined by Ω_m while T_{rec} is determined by D_b (aside from microphysical parameters). The near-coincidence between them traces to the fact that the baryon density is comparable to the matter density, $D_b \sim \Omega_m$. ~~This~~
 D_b is set by whatever baryogenesis mechanism sets $\eta \sim n_b/n_r \sim 10^{-10}$, while Ω_m may be due to annihilation of weakly interacting dark matter particles, two unrelated events. Some theorists have attempted to make models in which these two things are more closely related in their origins.

5.) To lowest approximation, the ${}^4\text{He}$ mass fraction yield predicted from primordial nucleosynthesis is

$$X_4 = 2X_n(t_{\text{BBN}}) = 2X_n(t_F) e^{-t_{\text{BBN}}/\tau_n}$$

where $X_n = \frac{(n_n/n_p)}{1 + (n_n/n_p)}$ and τ_n = neutron lifetime, t_{BBN} = time when $D \rightarrow {}^4\text{He}$.

and $(n_n/n_p)_{t_F} = e^{-Q/T_F}$.

Thus, in general, we have $\boxed{\frac{\delta X_4}{X_4} = \frac{\delta X_n}{X_n}}$, and

$\boxed{\frac{\delta X_n}{\delta(n_n/n_p)_F} = \left(1 + \left(\frac{n_n}{n_p}\right)_F\right)^{-2}}$, so that

$$\boxed{\frac{\delta X_n}{X_n} = \frac{1}{1 + \left(\frac{n_n}{n_p}\right)_F} \left(\frac{\delta(n_n/n_p)_F}{\left(\frac{n_n}{n_p}\right)_F} \right)}$$

for small changes in the neutron to proton ratio.

A change in $g_*(T)$, e.g., due to an additional light neutrino species, affects the ${}^4\text{He}$ yield in two ways:

- (i) The freeze-out temp. T_F of n_n/n_p changes: recall the expansion rate $H \sim g_*^{1/2} T^2$, and the weak interaction rate for $n \leftrightarrow p$ equilibration is $T_{\text{weak}} \sim T^5$. Freeze-out occurs when $T_{\text{weak}} \approx H$, which implies $g_*^{1/2} T_F^2 \approx T_F^5$, so that $\boxed{T_F \propto g_*^{1/6}}$

~~An increase in g_*~~ An increase in g_* at the time of BBN leads to a faster expansion rate at fixed T , causing the weak interactions to freeze-out earlier, leading to a higher $\left(\frac{n_n}{n_p}\right)_F$ and thus more ^4He .

To quantify this we need to relate $\delta\left(\frac{n_n}{n_p}\right)_F$ to δT_F :

Since $\left(\frac{n_n}{n_p}\right)_F = e^{-Q/T_F}$, we have

$$\delta\left(\left(\frac{n_n}{n_p}\right)_F\right) = \frac{Q}{T_F^2} e^{-Q/T_F} \delta T_F$$

Thus

$$\frac{\delta\left(\frac{n_n}{n_p}\right)_F}{\left(\frac{n_n}{n_p}\right)_F} = \frac{Q}{T_F} \frac{\delta T_F}{T_F}$$

Now since $T_F \propto g_*^{1/6}$, we have

$$\frac{\delta X_4}{X_4} = \frac{\delta X_n}{X_n} = \frac{1}{1 + \left(\frac{n_n}{n_p}\right)_F} \left(\frac{Q}{T_F} \right) \left(\frac{1}{6} \frac{\delta g_*}{g_*} \right)$$

(i) The time at which nucleosynthesis occurs changes: recall the relation between expansion time and temperature,

$$t \sim g_*^{-1/2} T^{-2}$$

Since the temperature at which BBN occurs (when the D bottleneck breaks) is fixed by the condition of nuclear statistical equilibrium, the time at

$$\text{which BBN starts scales as } t_{\text{BBN}} \sim g_*^{-1/2}, \text{ so } \frac{\delta t_{\text{BBN}}}{t_{\text{BBN}}} = -\frac{1}{2} \frac{\delta g_*}{g_*}$$

Since $X_n \propto e^{-t_{BBN}/\tau_n}$, we have

$$\frac{\delta X_n}{X_n} = -\frac{t_{BBN}}{\tau_n} \frac{S t_{BBN}}{t_{BBN}} = \frac{1}{2} \frac{S g_*}{g_*} \frac{t_{BBN}}{\tau_n}$$

Physically, increasing g_* \Rightarrow faster expansion rate at fixed $T \Rightarrow D$ bottleneck breaks earlier \Rightarrow more undecayed neutrons around \rightarrow more ^4He .

Putting these two pieces together, we have

$$\left| \begin{aligned} \frac{\delta X_4}{X_4} &= \frac{S g_*}{g_*} \left[\frac{1}{1 + \left(\frac{u_n}{v_p}\right)_F} \frac{Q}{T_F} \left(\frac{1}{6}\right) + \frac{1}{2} \frac{t_{BBN}}{\tau_n} \right] \\ &= 0.33 \frac{S g_*}{g_*} \end{aligned} \right.$$

where I've used the fiducial values

$$Q = 1.293 \text{ MeV}, T_F = 0.8 \text{ MeV}, t_{BBN} = 180 \text{ sec},$$

$$\text{and } \tau_n = 887 \text{ sec in the last line above.}$$

b.) Assume that neutrinos freeze-out before BBN (a reasonable but not exact assumption, since $T_F^{(v)} \approx 1 \text{ MeV}$ and $T_{BBN} = 0.8 \text{ MeV}$).

Then, as we worked out in class the contribution to

$$g_* \text{ per neutrino species is } S g_*^{(v)} = 2 \left(\frac{7}{8}\right) \left(\frac{T_F}{T_\gamma}\right)^4 S N_r$$

where the factor 2 is for the # of helicity states,
 $\frac{7}{8}$ is for fermions, and $\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$ ~~also~~

assuming neutrinos freeze out before e^+e^- annihilation
 (as we worked out in class). Thus,

$$\delta g_{\pm}(\nu) = 0.454 \delta N_\nu = 0.454(N_\nu - 3)$$

Thus,
$$\frac{\delta X_4}{X_4} = (0.33) \frac{(0.454)}{g_*} (N_\nu - 3)$$

Using the fiducial values $X_4 = 0.235$ and $g_* = 3.36$,

this gives
$$\frac{\delta X_4}{X_4} = 0.010 (N_\nu - 3)$$
. Thus an

extra neutrino species ($\delta N_\nu = 1$) increases the $^{4\text{He}}$ mass fraction
 from 0.235 to approx. 0.245.