Lecture 8:
General Relativity in 2 Lectures

Physical Principles

Principle of Equivalence between gravitation + inertia

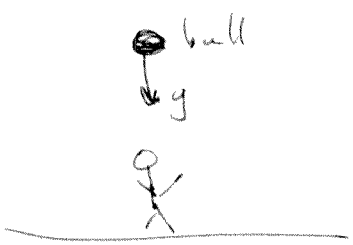
Galileo (Tower of Pisa): all objects fall in grav. field with observed same acceleration g. This explained by

Newton: $\underset{\text{inertia}}{F = ma}$ $\quad \underset{\text{gravity}}{F_g = \frac{GMm}{R^2}}$ $\Rightarrow a = g = \frac{GM}{R}$ indep. of m.

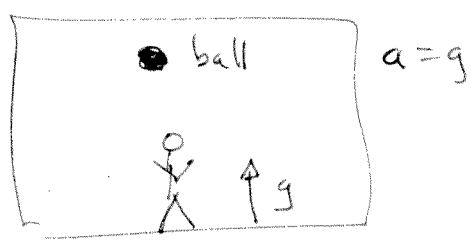
i.e. equivalence of inertial + gravitational mass.

Einstein: This is a deep principle, because it implies that a freely falling observer feels no gravitational force. (Struck him one day at the Swiss patent office.) $\Rightarrow$ Transcends Newton's laws.

E's Thought-experiment: (mutatis mutandem) Consider 2 observers:



Earth (flat)



Elevator in outer space accelerated at a=g

According to P.O.E., observer 2 feels, same physical effects as observer 1. In particular, by observing motions of bodies, he can't tell which situation he's in. (for uniform gravitational field, i.e. for local measurements in a gravitational field.) Observer in elevator could mistakenly believe he's at rest on earth and that earth makes ball hit him.

Corollary: if you're in a closed elevator on earth and somebody cuts supporting cord, you go into free fall. If you then carefully release a baseball, what happens? It stays at rest relative to you. ∴ By observing objects' motions, you wouldn't think you were in a grav. field at all: objects you release remain at rest (or move at const. velocity) w.r.t. you, instead of ~~falling at~~ accelerating at rate $\vec{a} = \vec{g}$ they way they do in earth's field. ∴ In this sense, the observable effects of uniform grav. field can be "turned off" by accelerating at $\vec{a} = \vec{g}$. This demonstrates local equivalence of inertia + (uniform) gravity.

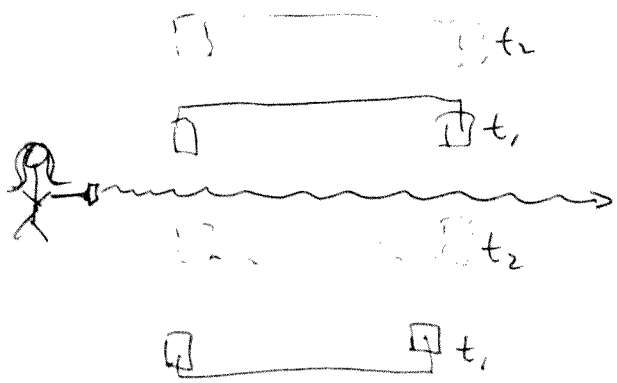
the freely falling elevator is then also inertial frame in sense of S.R. but since earth's field not uniform it is only local

IF ... a gl...

IF

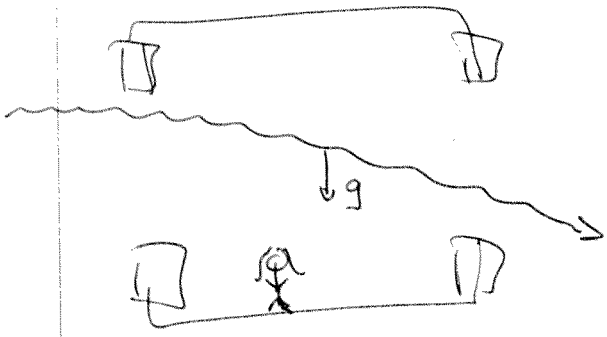
Consequences of P.O.E.

Consider 2nd thought-expt.: woman at rest shines flashlight into passing elevator accelerating upward at $\vec{a} = \vec{g}$. According to her, the inertial observer, light travels in a straight line.



Light enters at top left window at t_1 and emerges at bottom right window at t_2. She ascribes this to elevator's motion.

Now consider from pt. of view of ~~[illegible]~~ the man in elevator:

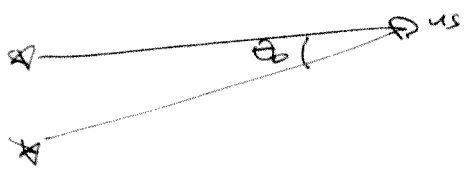


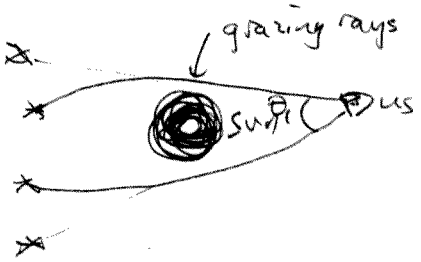
~~[illegible]~~ He sees light bend downward as it crosses elevator. But, since he's accel. upward at $\vec{g}$, he also believes he's at rest on earth's surface (by P.O.E.), so he thinks that light has been bent into a curved path by gravity!

But, by P.O.E., he's right: both descriptions are equally admissable, i.e., physically indistinguishable.

Therefore, P.O.E. implies gravitational deflection of light by massive bodies. Predicted correctly by E. in '16, verified in Eddington's African eclipse expedition in 1919.

2 stars





Newton: would say grav. force bends the light ray.

Einstein: this is unnatural. Such bending happens to all objects: light rays, particles, etc. Also, prefer to think light travels in straight line, along shortest dist. between 2 pts = geodesics (as in SR). ⟹ Reconcile these by the notion that spacetime itself is curved, distorted by presence of matter (mass-energy-momentum). i.e. gravity is curved spacetime. ← this piece inspired

Rubber sheet (pillow) analogy: gravity curves ST like a bowling ball deforms a pillow.

Einstein: 1.) curved ST tells mass-energy how to move (on geodesics of curved ST)

2.) Mass-energy tells ST how to curve (Einstein's eqns)

(a bonus: 1.) follows from (2)).

Since gravity is a property of ST itself, naturally all objects fall w/ same acceleration in a grav. field. Freely falling bodies experience no grav. "forces", they just are following their "natural" trajectories in curved ST. ∴ Geometrical gravity ⇔ P.O.E. if it applies to everything that it can be a property of spacetime (geometry) itself: Einstein's insight

~~How incorporate P.O.E. into language of physics? By the~~

~~Principle of General Covariance~~: Thus, can always locally turn off the effects of gravity by choosing a different reference frame, i.e., a locally inertial coordinate system, such that, w/in a small region, the laws of physics have same form as in unaccelerated Cartesian coord. systems in absence of gravity (i.e. we can "cancel" gravity by inertial forces i.e., in the Loc. Inertial frame, special relativity holds.

~~Weak P.O.E.~~:

Strong P.O.E.: "laws of physics" above means all laws of nature.

Weak P.O.E.: "laws of physics" above means "laws of motion of freely falling particles".

- For <u>curved</u> spacetime, Minkowski metric is replaced by more general metric $g_{\alpha\beta}$ that describes effects of gravity.

- Principle of Equivalence: <u>locally</u>, we can always find a coord. sys. where $g_{\alpha\beta}$ reduces to $\eta_{\alpha\beta}$ at a point. But we cannot do this <u>globally</u>: gravitational tidal effects are <u>not</u> equivalent to uniform acceleration.

- <u>Newton:</u>
  - <u>Gravity</u>: $\nabla^2 \varphi = 4\pi G \rho$

    Mass distribution gives rise to grav field

  - <u>Motion</u>: $\vec{F}_g = -\nabla \varphi = ma$

    2nd law tells particles how to accelerate in grav. field

<u>Einstein:</u>

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

stress-energy distribution tells spacetime how to curve

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu dx^\nu}{d\tau\, d\tau} = 0$$

Particles follow geodesics in curved spacetime

$\Gamma \sim$ first derivatives of $g_{\mu\nu}$.

- <u>Follows</u> from eqns of GR: elegant

- <u>Principle of Equivalence</u> $\Rightarrow$ <u>Princ. of General Covariance</u>:

  laws of physics must preserve their form under general coordinate transformations. $x^\mu \to x'^\mu$: required if they are to hold in a general gravitational field.

- Naturally leads to <u>tensors</u>: objects which transform in prescribed way under coord. transformations.
  - Tensor equations are <u>automatically</u> general-covariant: true in all coordinate frames.

Application : Geodesic EOM

effects of gravity on a system.

Consider particle moving freely (no other forces but gravity) in grav. field. By P.o.E., can make freely falling coordinates ξ^α s.t. EOM of particle is a straight-line in ST, i.e.

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \qquad (F=0)$$

where the proper time $d\tau$ is given by

$$d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

where $\eta_{\alpha\beta}$ = Minkowski metric $= \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ ← local inertial frame = freely falling frame

In SR, $d\tau^2$ is invariant under Lorentz transformations (~~generalized~~ lack of simultaneity): they agree on $d\tau^2$

Now construct any other coord. sys. x^μ, e.g. a Cartesian sys. at rest in lab, but may also be curvilinear, etc. Then $\xi^\alpha = \xi^\alpha(x^\mu)$ and EOM becomes

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = 0$$

(chain rule)

$$\Rightarrow \quad \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Mult. by $\frac{\partial x^\lambda}{\partial\xi^\alpha}$ and use $\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \delta^\lambda_\mu$ (product rule) to get

$$\frac{d^2x^\lambda}{d\tau^2} + \underbrace{\frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}}_{\Gamma^\lambda_{\mu\nu}\ \text{affine connection}}\frac{dx^\mu dx^\nu}{d\tau\, d\tau} = 0$$

∴ Geodesic EOM in general coord. sys. is $\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu dx^\nu}{d\tau\, d\tau} = 0}$

~~GR (cont'd) will follow Weinberg some but also give more geometric inte~~

~~N.B.: last time~~ PDE → can locally turn off effects of grav. field ⟺ choosing a local inertial frame = freely falling observer, carrying coords. ξ^α s.t. particle (in the grav. field) satisfies $\frac{d^2\xi^\alpha}{d\tau^2}=0$

where $d\tau^2$ = invariant interval of SR

$$= \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

(not $dx^\mu dx^\nu$)

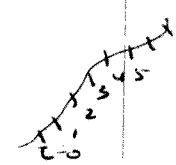
Minkowski metric

Transform to arbitrary coords. $\xi^\alpha \to x^\mu$.

→ geodesic eqn.: $$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu dx^\nu}{d\tau\, d\tau}=0$$

where τ = affine parameter along the geodesic curve

= ~~measure of~~ counting along the curve



In the new coords., the invariant interval is

$$d\tau^2 = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu$$

where $g_{\mu\nu}$ = metric tensor $= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$

This is EOM of "free" particle in curved spacetime, i.e.

force of gravity, $\ddot{\vec{x}} = -\frac{GMm}{r^2}\hat{x}$ has been replaced

by spacetime curvature

$\therefore$ gravitational field : $g_{\mu\nu} \neq \eta_{\mu\nu}$ globally $\Rightarrow \Gamma^{\lambda}_{\mu\nu} \neq 0$

$\Updownarrow$

Can show geodesic equ. reduces to Newton's 2nd law

for weak grav. fields + slow particles. (Homework?)

$(v \ll c)$

Also, have $d\tau^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$

$$\equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

where $g_{\mu\nu}$ = <u>metric tensor</u> $= \frac{\partial \xi^\alpha \partial \xi^\beta}{\partial x^\mu \partial x^\nu} \eta_{\alpha\beta}$.

<u>Note</u>: values of $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ at a pt. X in the arbitrary coord. sys. x^μ give enough info. to determine the locally inertial coords. $\xi^\alpha(x)$ in a neighborhood of X. Since grav. field has no effect in the LIF, it makes sense that <u>all</u> effects of gravitation are comprised in Γ and g.

(Later, we'll derive relation between Γ and g.)

Note SR vs. GR: In SR, have <u>global</u> inertial frames; in GR, due to gravity (curvature) can in general only construct <u>local</u> IF, i.e. can transform $g_{\mu\nu} \to \eta_{\mu\nu}$ only at a pt., but in general must make diff. transformation at different pts. (POE holds only <u>locally</u> in grav. field, observers totally accel. away from each other)

This method of starting in LIF + transforming is cumbersome. Instead, Incorporate P.O.E. into physics via <u>Princ. of General Covariance</u>:

A physical eqn. holds in a general grav. field if:

a.) it holds in absence of gravity, i.e. it agrees w/ S.R. when take limits $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\Gamma^\mu_{\nu\sigma} \to 0$.

b.) it is <u>generally covariant</u>: i.e. preserves its form under a general coord. transf. $x \to x'$. (e.g. $x \to \xi$). ← <u>General</u> Relativity

Suppose are in arbitrary grav. field and consider an eqn. satisfying (a) and (b). From (b.), it is true in all coord. systems if true in 1. But at any pt. X can construct LIF's in which gravity is absent. Then (a.) tells us the eqn. holds in these LIF's, and thus in all other coord. systems.

∴ Gen. covariance ⟺ eqn. true in grav. field if true in absence of gravity.

Note: Gen. cov. is not an invariance principle like Lorentz invariance, but is statement about effects of gravity. It's more like a local gauge invariance (like EM).

∴ Construct eqns. invariant under gen. coord. transformations.

(note: in SR, have preferred inertial frames, but in grav. field, GIF's don't exist: observers tidally accelerated toward/away from each other.)

The way to guarantee Gen. Covariance (i.e. form invariant eqns. under $x \to x'$), is to only allow tensor eqns. : Tensors are objects w/ definite properties under coord. transformations. So I'll now briefly review tensors:

Tensor Analysis

Scalars:

$x^\mu \to x'^\mu \qquad \varphi'(x'^\mu) = \varphi(x^\mu)$ invariant under general coord. transf.

4-Vectors:

e.g. $x^\mu(x'^\mu) = f^\mu(x'^\mu)$

e.g. $dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$ (summation convention here + throughout)

Generalize: $A^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu$ 4-vector

$\partial\varphi/\partial x^\mu = \frac{\partial \varphi}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu}$

Also: $A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu$ 1-form

$A^\mu B_\mu = A'^\mu B'_\mu$ is invariant (scalar product) (summation convention)

Tensors: $A^{\mu\nu}$ transforms like product of vectors:

$$A^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} A'^{\rho\sigma}$$

Similarly $A_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} A'_{\rho\sigma}$

Again, scalar products are invariant. Inverse: $A_{\mu\nu}(A^{-1})^{\nu\sigma} = \delta^\sigma_\mu$

Metric tensor: $g_{\mu\nu}$.

invariant distance: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

scalar product, invariant under gen. coord. transf.

$A_\mu = g_{\mu\nu} A^\nu$ represents same physical quantity

Tensor Derivatives:

$$A'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}(x)$$

Now, $\frac{\partial}{\partial x'^{\lambda}} = \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial}{\partial x^{\rho}}$ is transform of derivative.

Thus

$$\frac{\partial A'^{\mu}(x')}{\partial x'^{\lambda}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial A^{\nu}(x)}{\partial x^{\rho}} + \underline{\frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left(\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \right) A^{\nu}(x)}$$

Taking derivatives of 2 different points x, x'.

extra term: derivative of a tensor is not (does not transform as) a tensor

- Add something proportional to A^{ν} which makes it transform as a tensor:

Covariant Derivative:

$$\nabla_{\lambda} A^{\mu} = \partial_{\lambda} A^{\mu} + \Gamma^{\mu}_{\lambda\nu} A^{\nu}$$ is a tensor.

$\Gamma^{\mu}_{\lambda\nu}$ = affine connection or Christoffel symbols, not a tensor itself. Γ symmetric in lower indices

More generally: $\nabla_{\rho} A^{\mu\nu}_{\sigma} = \partial_{\rho} A^{\mu\nu}_{\sigma} + \Gamma^{\mu}_{\rho\lambda} A^{\lambda\nu}_{\sigma} + \Gamma^{\nu}_{\rho\lambda} A^{\mu\lambda}_{\sigma} - \Gamma^{\lambda}_{\sigma\rho} A^{\mu\nu}_{\lambda}$

Γ and g:

$\nabla_{\nu} A_{\mu} = g_{\mu\rho} \nabla_{\nu} A^{\rho}$ assuming covar. diff. commutes w/ raising + lowering indices

but $\nabla_{\nu} A_{\mu} = \nabla_{\nu} (g_{\mu\rho} A^{\rho})$

$= g_{\mu\rho} \nabla_{\nu} A^{\rho} + (\nabla_{\nu} g_{\mu\rho}) A^{\rho} \Rightarrow \boxed{\nabla_{\nu} g_{\mu\rho} = 0}$

Covariant [illegible]

- This makes sense: can locally choose coords. ($x \to x'$) s.t. $g_{\mu\nu} \to \eta_{\mu\nu}$ at a point. In that frame,

$$\Gamma^{\lambda}_{\mu\nu} = 0 = \partial_\lambda g_{\mu\nu} \Rightarrow \nabla_\lambda g_{\mu\nu} = 0$$

Since it is a tensor equ., it must hold in <u>all</u> frames.

$$\nabla_\lambda g_{\mu\nu} = 0 = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^{K}_{\lambda\mu} g_{K\nu} - \Gamma^{K}_{\lambda\nu} g_{K\mu}$$

Add same equ. w/ $\mu \leftrightarrow \lambda$ interchanged and subtract w/ $\nu \leftrightarrow \lambda$ interchanged:

$$\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} = \Gamma^{K}_{\lambda\mu} g_{K\nu} + \Gamma^{K}_{\lambda\nu} g_{K\mu} + \Gamma^{K}_{\mu\lambda} g_{K\nu} + \Gamma^{K}_{\mu\nu} g_{\lambda K} - \Gamma^{K}_{\nu\mu} g_{K\lambda} - \Gamma^{K}_{\nu\lambda} g_{K\mu}$$

Use symmetries of $g_{\mu\nu} = g_{\nu\mu}$ and $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$:

$$\text{RHS} = 2 g_{K\nu} \Gamma^{K}_{\lambda\mu}$$

Mult. by $g^{\nu\sigma}$ and use $g^{\nu\sigma} g_{K\nu} = \delta^{\sigma}_{K}$ to find:

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right]$$

- In absence of gravity, $g \to \eta$ <u>globally</u>, $\Gamma \to 0$, $\nabla_\lambda \to \partial_\lambda$.
- Can make flat-ST equs. applicable in grav. field by replacing ∂ by ∇.
- <u>Example</u>: energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0 \to \nabla_\mu T^{\mu\nu} = 0$