Crazy Theorems and Conjectures

Wopat #289
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Outline

• Gödel's incompleteness theorems
• Continuum hypothesis
• Riemann hypothesis
• 3n +1 Problem
• 4 color problem
Gödel’s Incompleteness Theorems

• Some definitions
  – A theory = set of sentences expressed in formal language
  – Axiom = sentence that is assumed true without proof
  – Theorem = sentence that is implied by axioms
  – Effectively generated theory = a computer program could list all the axioms of the theory
    • i.e. maybe there are a finite number of axioms, or maybe there is an infinite number of axioms that could be generated in a well defined way
Gödel’s Incompleteness Theorems

• A set of axioms is **complete** if every sentence that can be written can be proved or disproved from axioms

• A set of axioms is **consistent** if there is no statement that can be both proved and disproved from axioms
Gödel’s Incompleteness Theorems

• A set of axioms is **complete** if every sentence that can be written can be proved or disproved from axioms

• A set of axioms is **consistent** if there is no statement that can be both proved and disproved from axioms

• **Godel’s 1st Incompleteness theorem:**
  
  An effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete
Gödel’s Incompleteness Theorems

• What does Godel’s 1st Incompleteness theorem mean?
  – For any theory that includes basic arithmetic, there will be theorems that are true but can’t be proven OH MY GOD
Gödel’s Incompleteness Theorems

• Sketch of proof:
  1. Assume that we can label propositions with natural numbers (Gödel devised a complicated way of doing this)
  2. Consider propositions that are functions of a single variable, P_n(w) (n denotes which proposition)
  3. Let A_n denote the nth proof
  4. Consider proposition P_k(w): “there does not exist x such that A_x proves P_w(w)”
  5. What about P_k(k)? Asserts that “there is no proof of P_k(k)”
     – There cannot be a proof of P_k(k) because if there were, then P_k(k) would be false and this would mean that the theory is inconsistent
     – But that means that P_k(k) is true! So it is true, but unprovable
Gödel’s Incompleteness Theorems

• 2nd Incompleteness Theorem:
  – If a theory contains basic arithmetic, then one sentence it cannot prove is the consistency of the system itself
Gödel’s Incompleteness Theorems

• Some implications:
  – No axiomatic system can prove all possible truths while proving no falsehoods
  
  – The halting problem is unsolvable: there is no program that can determine whether another program will eventually halt
  
  – Continuum hypothesis…
Continuum Hypothesis

- There are different levels of ‘infinity’
  - E.g. there are more real numbers than natural numbers
  - Cardinality of ‘countable infinity’ = $\aleph_0$
  - Cardinality of continuum = $2^{(\aleph_0)} = c$
  - Cardinality of set of all functions = $2^{(2^{\aleph_0})}$
  - ‘Set of all subsets of a set’
Continuum Hypothesis

• Continuum hypothesis: there is no infinity between $\kappa_0$ and $2^{\kappa_0}$
  – i.e. there is no set $S$ for which $\aleph_0 < |S| < 2^{\aleph_0}$.

• In 1940, Kurt Godel shows that the continuum hypothesis cannot be disproved from standard axioms of set theory
The Riemann Hypothesis

- The Riemann zeta function
  - For Re(s) > 1,
    \[ \zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} - \ldots \]
  - For all s,
    \[ \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s) \]
The Riemann Hypothesis

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    \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)
    \]
    \[
    \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt
    \]
The Riemann Hypothesis

• The hypothesis: all non-trivial zeros, $s_0$, of the zeta function have $\text{Re}(s_0) = 1/2$ (Riemann, 1859)

• Has been verified numerically for first 10 trillion zeros
The Zeta Function

![Diagram of the Zeta Function](image)

- **Trivial zeros**
- **Non-trivial zeros**
- **Pole**
- **Critical line**

The diagram illustrates the behavior of the Zeta function in the complex plane, highlighting key features such as the trivial zeros and the critical line.
The Riemann Hypothesis

• Relationship of the zeta function to primes
  – For Re(s) > 1

\[ \zeta(s) = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{7^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots \]

  – More importantly,

\[ \pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(x^{1/n}) = \text{number of primes less than } x \]
The Riemann Hypothesis

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    \]
    \[
    \mu(n) = \begin{cases} 
    1 & \text{if prime factorization of } n \text{ has even number of terms with no repeats} \\
    -1 & \text{if prime factorization of } n \text{ has odd number of terms with no repeats} \\
    0 & \text{if prime factorization of } n \text{ has repeats}
    \end{cases}
    \]
The Riemann Hypothesis

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    \]
    \[
    \Pi(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}) - \ln(2) + \int_{x}^{\infty} \frac{dt}{t(t^2 - 1)\ln(t)}
    \]
The Riemann Hypothesis

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    \]
    \[
    \text{Li} = \int_{0}^{x} \frac{dt}{\ln(t)}
    \]
The Riemann Hypothesis

• Relationship of the zeta function to primes
  – For Re(s) > 1
    \[ \zeta(s) = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{7^{-s}} \cdots \frac{1}{1 - p^{-s}} \cdots \]
  – More importantly,
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    \pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \prod(x^{1/n}) = \text{number of primes less than } x
    \]
    \[
    \Pi(x) = Li(x) + \sum_{\rho} Li(x^\rho) - \ln(2) + \int_{x}^{\infty} \frac{dt}{t(t^2 - 1)\ln(t)}
    \]
    \text{SUM OVER NON-TRIVIAL ZEROS OF ZETA FUNCTION}
Riemann Hypothesis

Why do we care?

- The hypothesis has implications for the distribution of prime numbers
  - The oscillations of prime numbers around their ‘expected positions’ controlled by zeros of zeta function
  - A number of interesting results and related theorems, such as
    \[
    |\pi(x) - Li(x)| < \frac{1}{8\pi} \sqrt{x \ln(x)} \quad \text{for } x \geq 2657
    \]

- Zeros correspond to energy levels of a quantum chaotic system? CRAAAAAZY
The 3n+1 Problem (or Collatz conjecture)

• **The process**: take any natural number, n. If it is even, divide by 2. If it is odd, multiply by 3 and add 1.

• **The conjecture**: for any starting value of n, the sequence eventually reaches the loop 4, 2, 1, 4, 2, 1, 4, 2, 1 …, etc.

• Has been checked by computer up to starting values of $\sim5\times10^{18}$
Steps to reach (4,2,1) sequence as a function of n
Directed graph showing progression of sequence
Directed graph showing progression of sequence
The 3n+1 problem

• No proof of conjecture exists
• An extension of the conjecture has been proved undecidable, but not the conjecture itself
• Can extend to complex numbers with

\[ f(z) = \frac{1}{2} z \cos^2\left(\frac{\pi}{2} z\right) + (3z + 1) \sin^2\left(\frac{\pi}{2} z\right) \]

\[ \rightarrow 3n+1 \text{ fractal} \]
The 4 Color Problem

• **The theorem:** 4 colors are sufficient to color any 2-D map so that no two adjacent regions have the same color

• Proved in 1976 by Appel and Haken

• First major theorem to be proven with a computer
A Counter Example???
NO
The 4 Color Problem

• How the proof worked:
  – Proof by contradiction: if true, there must be a minimal counterexample. Show that such a minimal counterexample can’t exist.
  – Unavoidable set: every map must contain a configuration from this set
  – Reducible configuration: if it occurs in a map, then we can simplify the map while keeping the number of required colors the same.
  – Appel and Haken generated an unavoidable, reducible set of 1936 maps → minimal counterexample can’t exist
    • Maps checked on computer
THE END