


# Crazy Theorems and Conjectures

Wopat #289

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# Outline

- Gödel's incompleteness theorems
- Continuum hypothesis
- Riemann hypothesis
- 3n +1 Problem
- 4 color problem

# Gödel's Incompleteness Theorems

- Some definitions
  - A theory = set of sentences expressed in formal language
  - Axiom = sentence that is assumed true without proof
  - Theorem = sentence that is implied by axioms
  - Effectively generated theory = a computer program could list all the axioms of the theory
    - i.e. maybe there are a finite number of axioms, or maybe there is an infinite number of axioms that could be generated in a well defined way

# Gödel's Incompleteness Theorems

- A set of axioms is **complete** if every sentence that can be written can be proved or disproved from axioms
- A set of axioms is **consistent** if there is no statement that can be both proved and disproved from axioms

# Gödel’s Incompleteness Theorems

- A set of axioms is **complete** if every sentence that can be written can be proved or disproved from axioms
- A set of axioms is **consistent** if there is no statement that can be both proved and disproved from axioms
- **Godel’s 1st Incompleteness theorem:**

  **An effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete**

# Gödel's Incompleteness Theorems

- What does Godel's 1st Incompleteness theorem mean?
  - For any theory that includes basic arithmetic, there will be theorems that are true but can't be proven OH MY GOD

# Gödel's Incompleteness Theorems

- Sketch of proof:
  1. Assume that we can label propositions with natural numbers (Godel devised a complicated way of doing this)
  2. Consider propositions that are functions of a single variable, $P_n(w)$ (n denotes which proposition)
  3. Let $A_n$ denote the nth proof
  4. Consider proposition $P_k(w)$: "there does not exist x such that $A_x$ proves $P_w(w)$"
  5. What about $P_k(k)$? Asserts that "there is no proof of $P_k(k)$"
     - There cannot be a proof of $P_k(k)$ because if there were, then $P_K(k)$ would be false and this would mean that the theory is inconsistent
     - But that means that $P_k(k)$ is true! So it is true, but unprovable

# Gödel's Incompleteness Theorems

- **2nd Incompleteness Theorem:**
  - If a theory contains basic arithmetic, then one sentence it cannot prove is the consistency of the system itself

# Gödel's Incompleteness Theorems

- Some implications:
  - No axiomatic system can prove all possible truths while proving no falsehoods
  - The halting problem is unsolvable: there is no program that can determine whether another program will eventually halt
  - Continuum hypothesis…

# Continuum Hypothesis

- There are different levels of ‘infinity’
  - E.g. there are more real numbers than natural numbers
  - Cardinality of ‘countable infinity’ = $\aleph_0$
  - Cardinality of continuum = 2^($\aleph_0$)=c
  - Cardinality of set of all functions = 2^(2^ $\aleph_0$))
  - ‘Set of all subsets of a set’

| $\mathbb{N}$ | $\leftrightarrow$ | *reals in* (0,1) |
|---|---|---|
| 1 | $\leftrightarrow$ | .835987... |
| 2 | $\leftrightarrow$ | .250000... |
| 3 | $\leftrightarrow$ | .559423... |
| 4 | $\leftrightarrow$ | .500000... |
| 5 | $\leftrightarrow$ | .728532... |
| 6 | $\leftrightarrow$ | .845312... |
| $\vdots$ | | $\vdots$ |
| $n$ | $\leftrightarrow$ | $.r_1 r_2 r_3 r_4 r_5 \ldots r_n \ldots$ |
| $\vdots$ | | $\vdots$ |

# Continuum Hypothesis

- Continuum hypothesis: there is no infinity between $\aleph_0$ and 2^($\aleph_0$)
  - i.e. there is no set S for which $\aleph_0 < |S| < 2^{\aleph_0}$.

- In 1940, Kurt Godel shows that the continuum hypothesis cannot be disproved from standard axioms of set theory

# The Riemann Hypothesis

- The Riemann zeta function
  - For Re(s) > 1,

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} - \cdots$$

  - For all s,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

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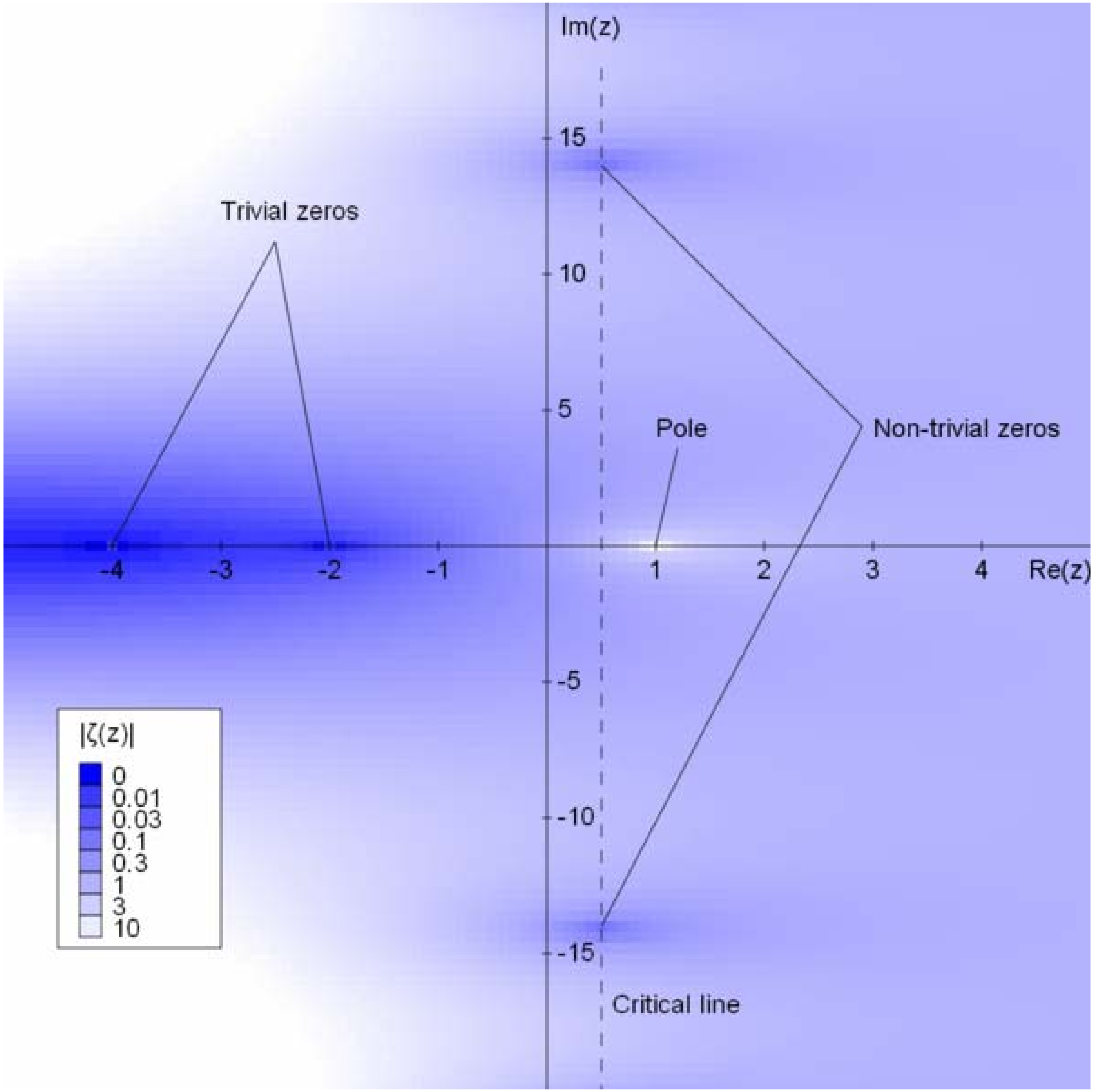
$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$$

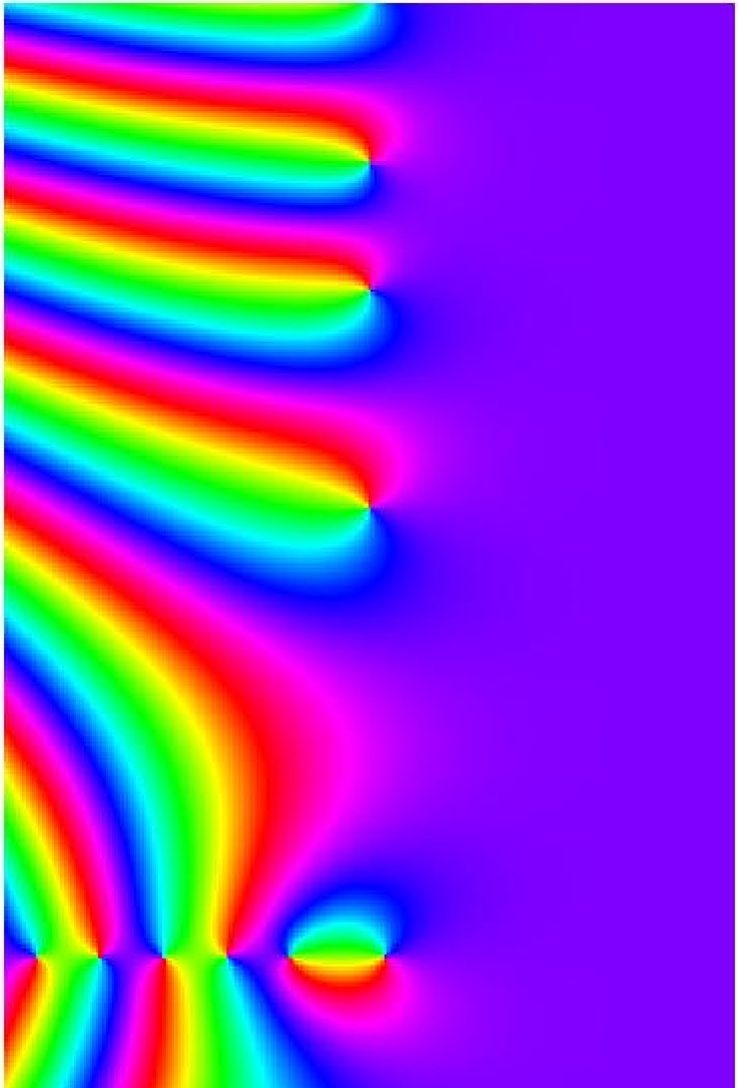
# The Riemann Hypothesis

- The hypothesis: **all non-trivial zeros, $s_0$, of the zeta function have $Re(s_0) = 1/2$**
  (Riemann, 1859)

- Has been verified numerically for first 10 trillion zeros

# The Zeta Function





# The Riemann Hypothesis

- Relationship of the zeta function to primes
  - For Re(s) > 1

$$\zeta(s) = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{7-^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots$$

  - More importantly,

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(x^{1/n}) = \text{number of primes less than x}$$

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$$\mu(n) = \begin{cases} \text{1 if prime factorization of n has even number of terms with no repeats} \\ \text{-1 if prime factorization of n has odd number of terms with no repeats} \\ \text{0 if prime factorization of n has repeats} \end{cases}$$

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$$\Pi(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \ln(2) + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\ln(t)}$$

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$$Li = \int_{0}^{x} \frac{dt}{\ln(t)}$$

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**SUM OVER NON-TRIVIAL ZEROS OF ZETA FUNCTION**

# Riemann Hypothesis

- Why do we care?
  - The hypothesis has implications for the distribution of prime numbers
    - The oscillations of prime numbers around their ‘expected positions’ controlled by zeros of zeta function
    - A number of interesting results and related theorems, such as

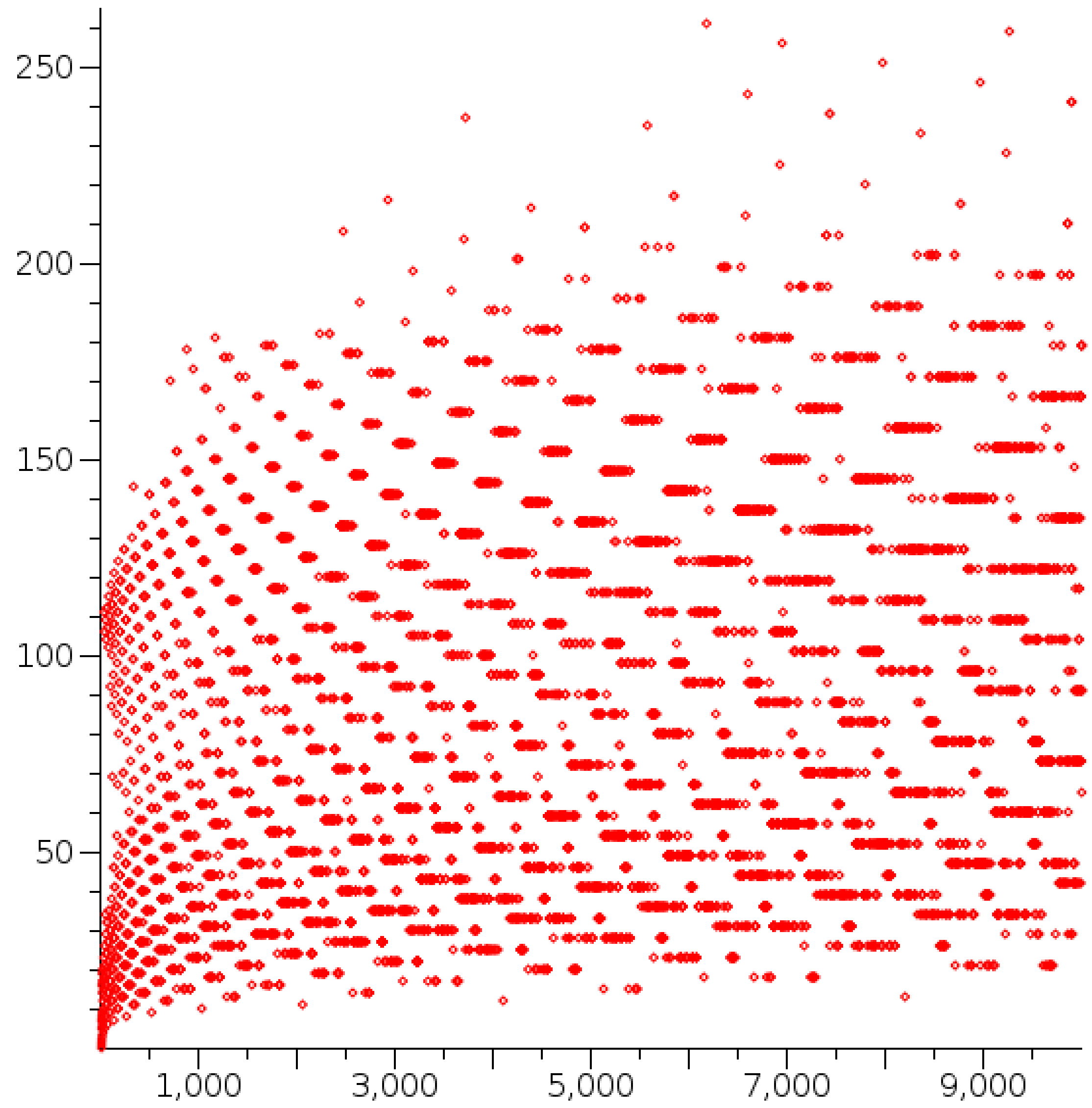
$$\left|\pi(x) - Li(x)\right| < \frac{1}{8\pi}\sqrt{x}\ln(x) \quad \text{for } x \geq 2657$$

    - http://www.aimath.org/WWN/rh/rh.pdf
    - Zeros correspond to energy levels of a quantum chaotic system? CRAAAAZY

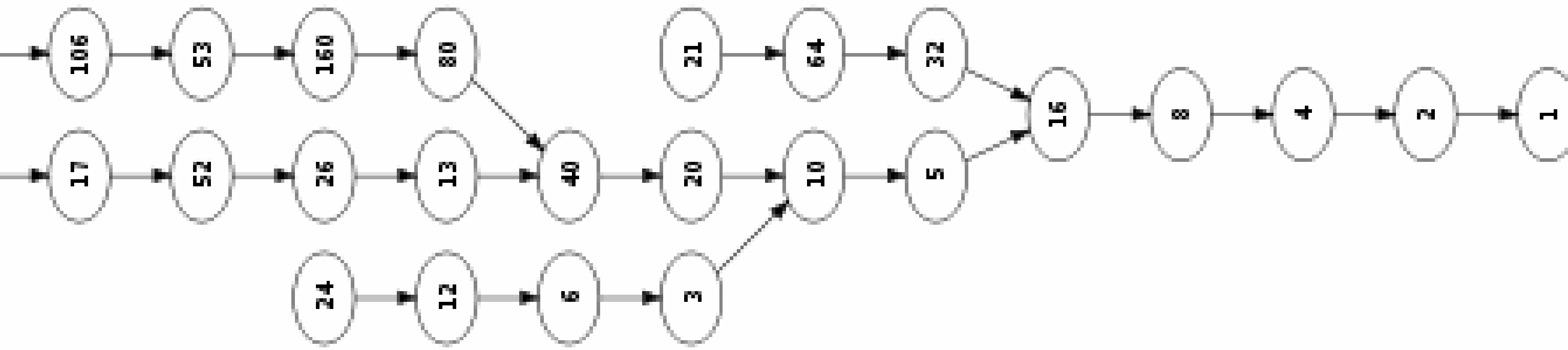
# The 3n+1 Problem (or Collatz conjecture)

- **The process**: take any natural number, n. If it is even, divide by 2. If it is odd, multiply by 3 and add 1.
- **The conjecture**: for any starting value of n, the sequence eventually reaches the loop 4, 2, 1, 4, 2, 1, 4, 2, 1 …, etc.
- Has been checked by computer up to starting values of ~$5 \times 10^{18}$

# Steps to reach (4,2,1) sequence as a function of n



# Directed graph showing progression of sequence



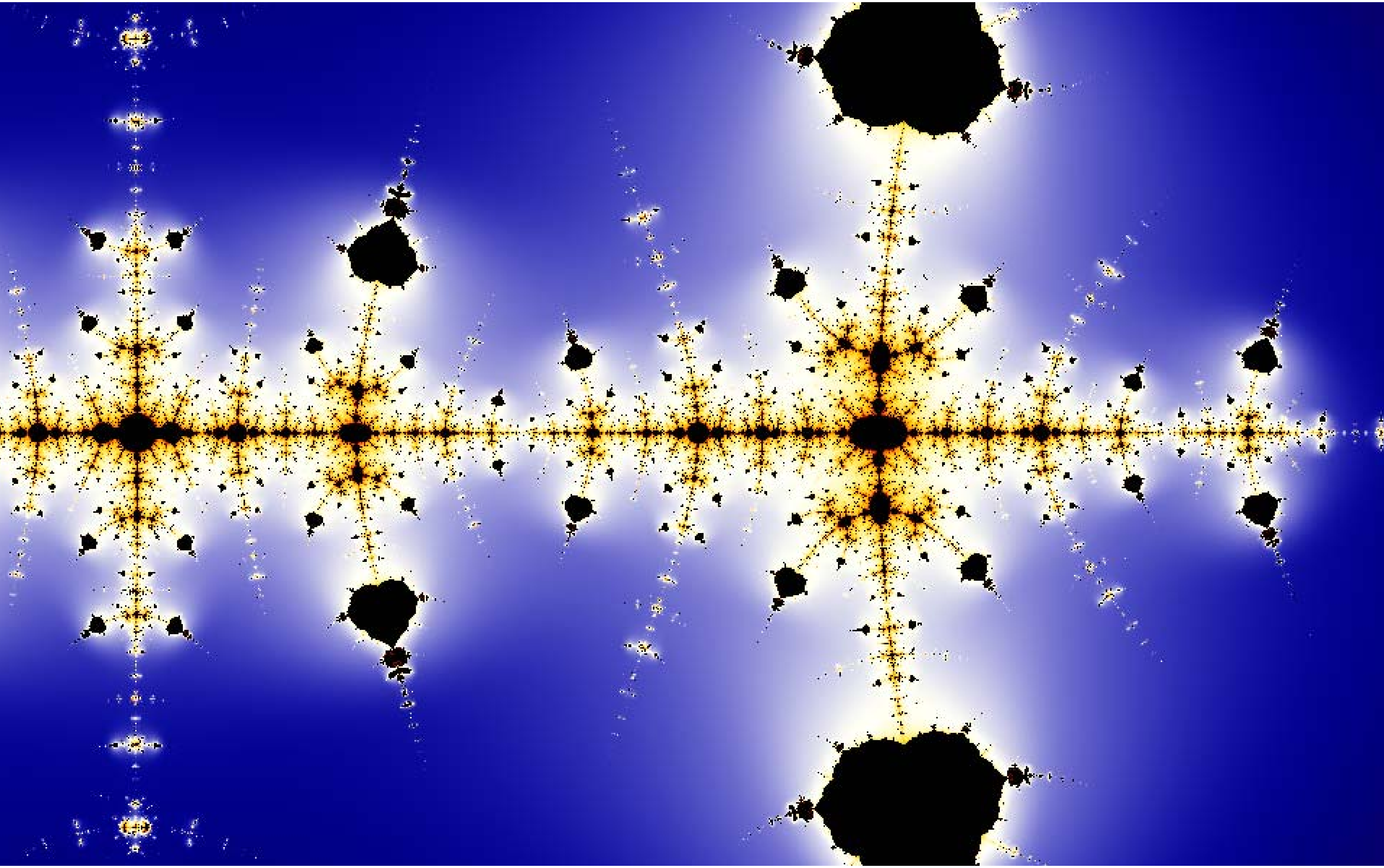
# Directed graph showing progression of sequence

# The 3n+1 problem

- No proof of conjecture exists
- An extension of the conjecture has been proved undecidable, but not the conjecture itself
- Can extend to complex numbers with

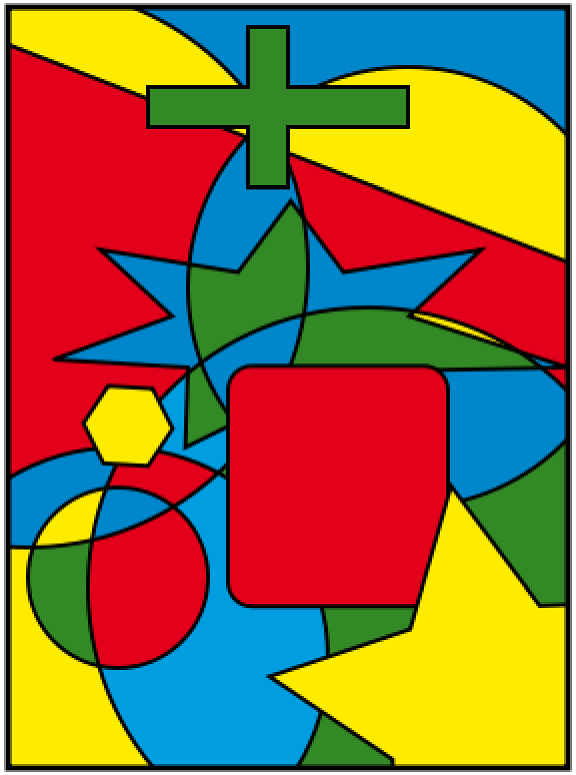
$$f(z) = \frac{1}{2} z \cos^2\left(\frac{\pi}{2} z\right) + (3z+1)\sin^2\left(\frac{\pi}{2} z\right)$$

→ 3n+1 fractal

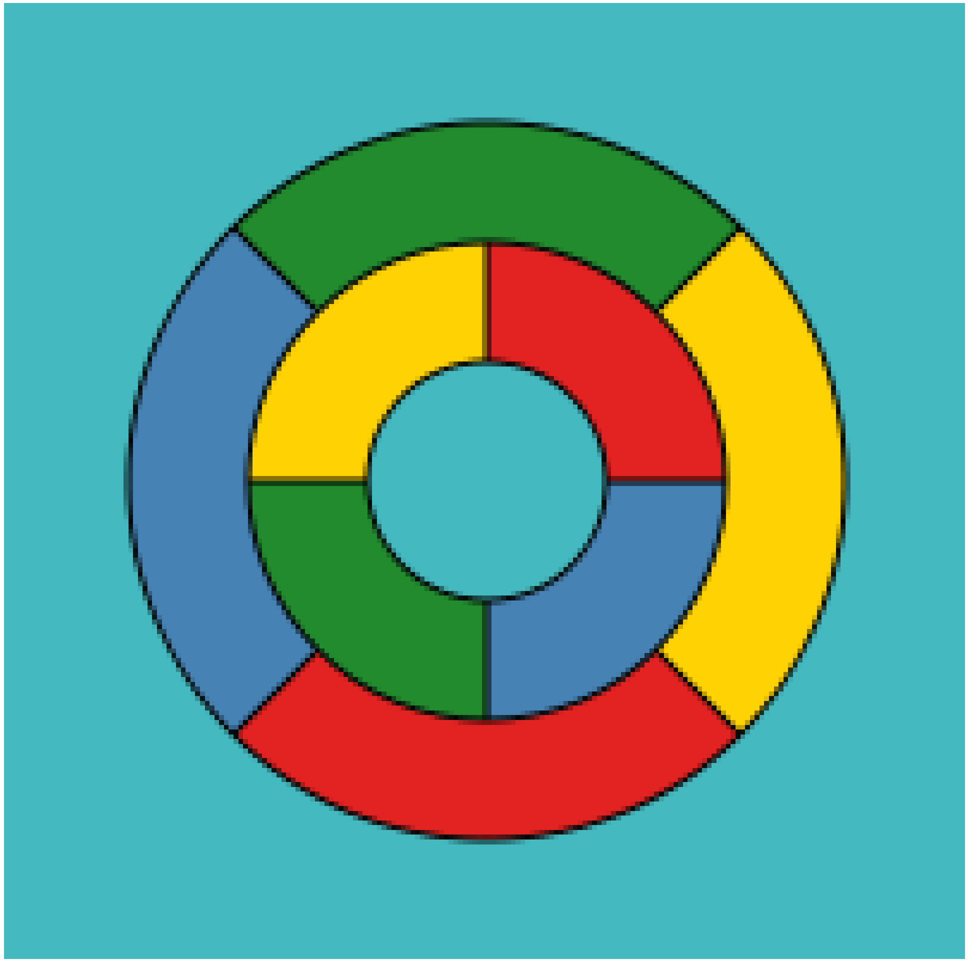


# The 4 Color Problem

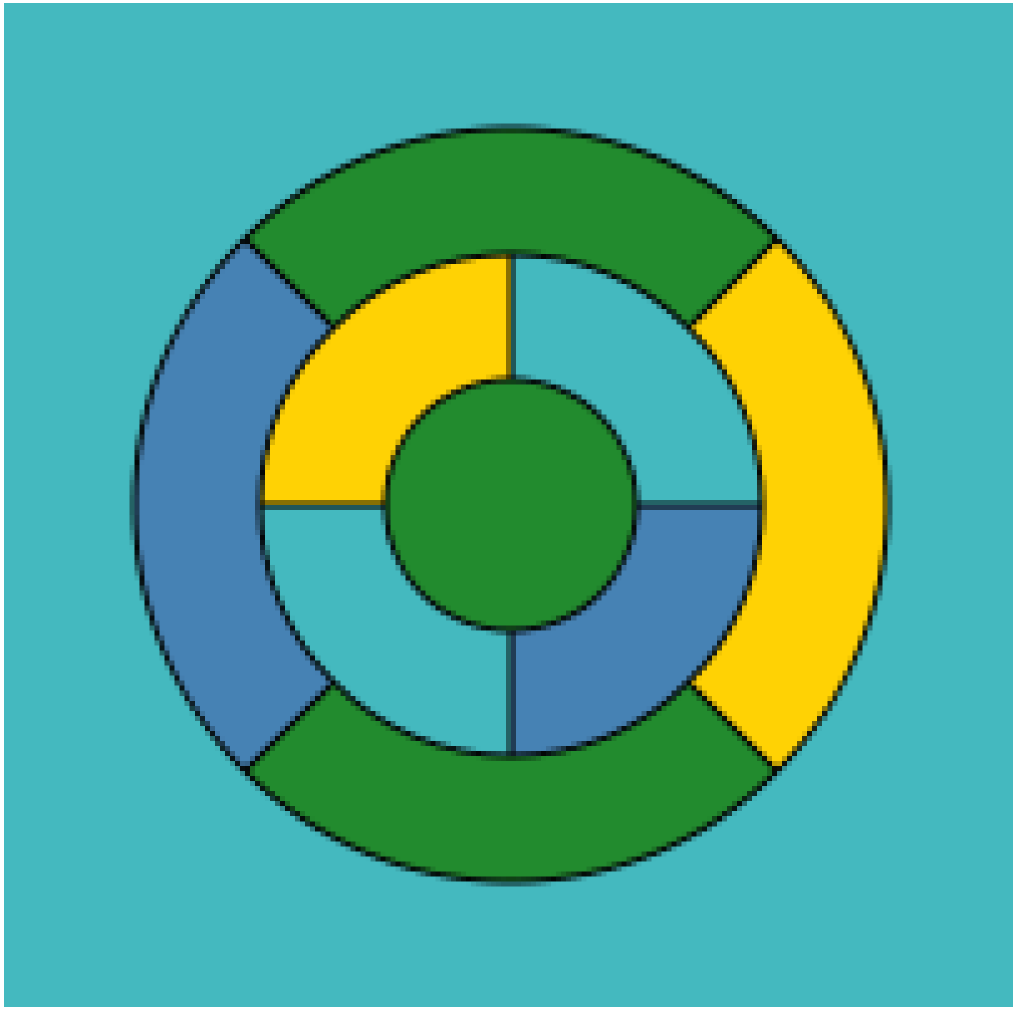
- **The theorem:** 4 colors are sufficient to color any 2-D map so that no two adjacent regions have the same color
- Proved in 1976 by Appel and Haken
- First major theorem to be proven with a computer



# A Counter Example???



NO



# The 4 Color Problem

- How the proof worked:
  - Proof by contradiction: if true, there must be a minimal counterexample. Show that such a minimal counter example can't exist.
  - Unavoidable set: every map must contain a configuration from this set
  - Reducible configuration: if it occurs in a map, then we can simplify the map while keeping the number of required colors the same.
  - Appel and Haken generated an unavoidable, reducible set of 1936 maps → minimal counter example can't exist
    - Maps checked on computer

THE END