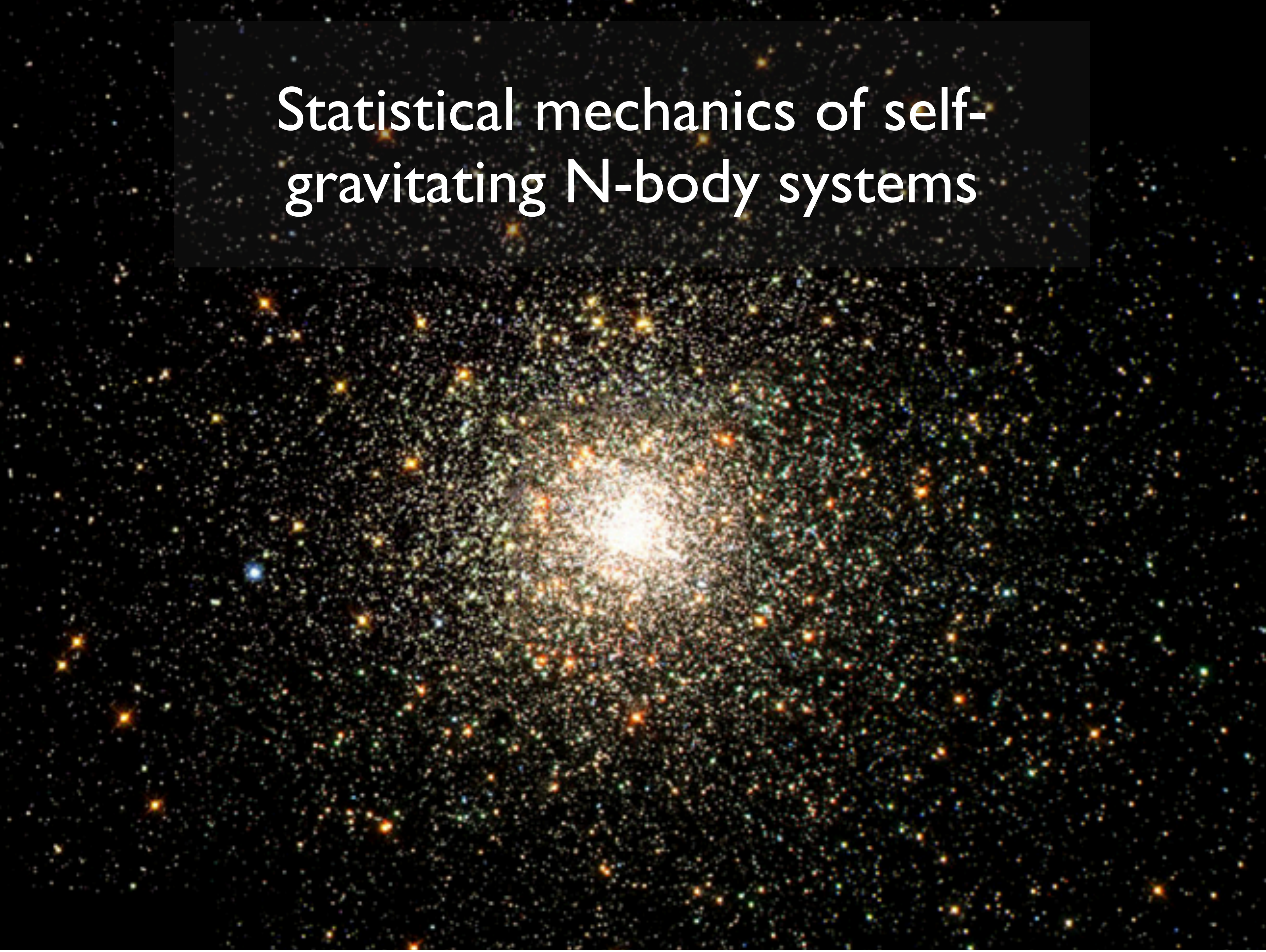
Statistical mechanics of self-gravitating N-body systems



gas in a box	stellar system
molecules, m ~ 10^{-24} g	stars, m ~ 10^{33} g WIMPs, m ~ 10^{-22} g
N ~ 10^{23}	N ~10^5 (globular clusters), ~10^5-10^{11} (stars in galaxies), ~10^{65} (WIMPs in galaxies)
short-range forces	long-range forces (gravity)
confined in a box	confined by self-gravity
mean free path << system size (Knudsen number Kn << 1)	mean free path >> system size (Kn >> 1)

K = kinetic energy
W = potential energy
E = $K+W$ = total energy

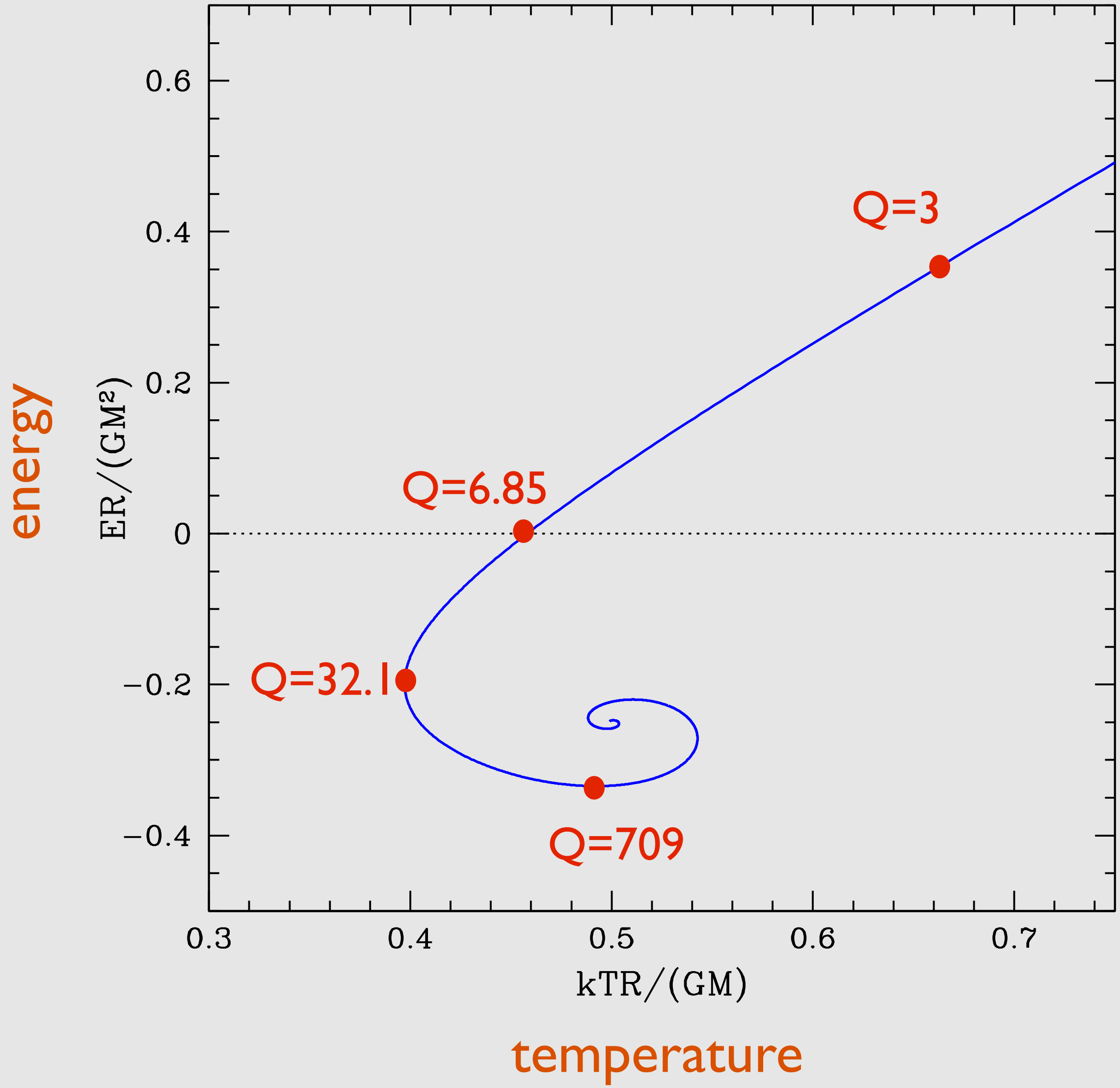
virial theorem states that in a steady state

$$2K + W = E + K = 0 \quad \text{or} \quad E = -K.$$

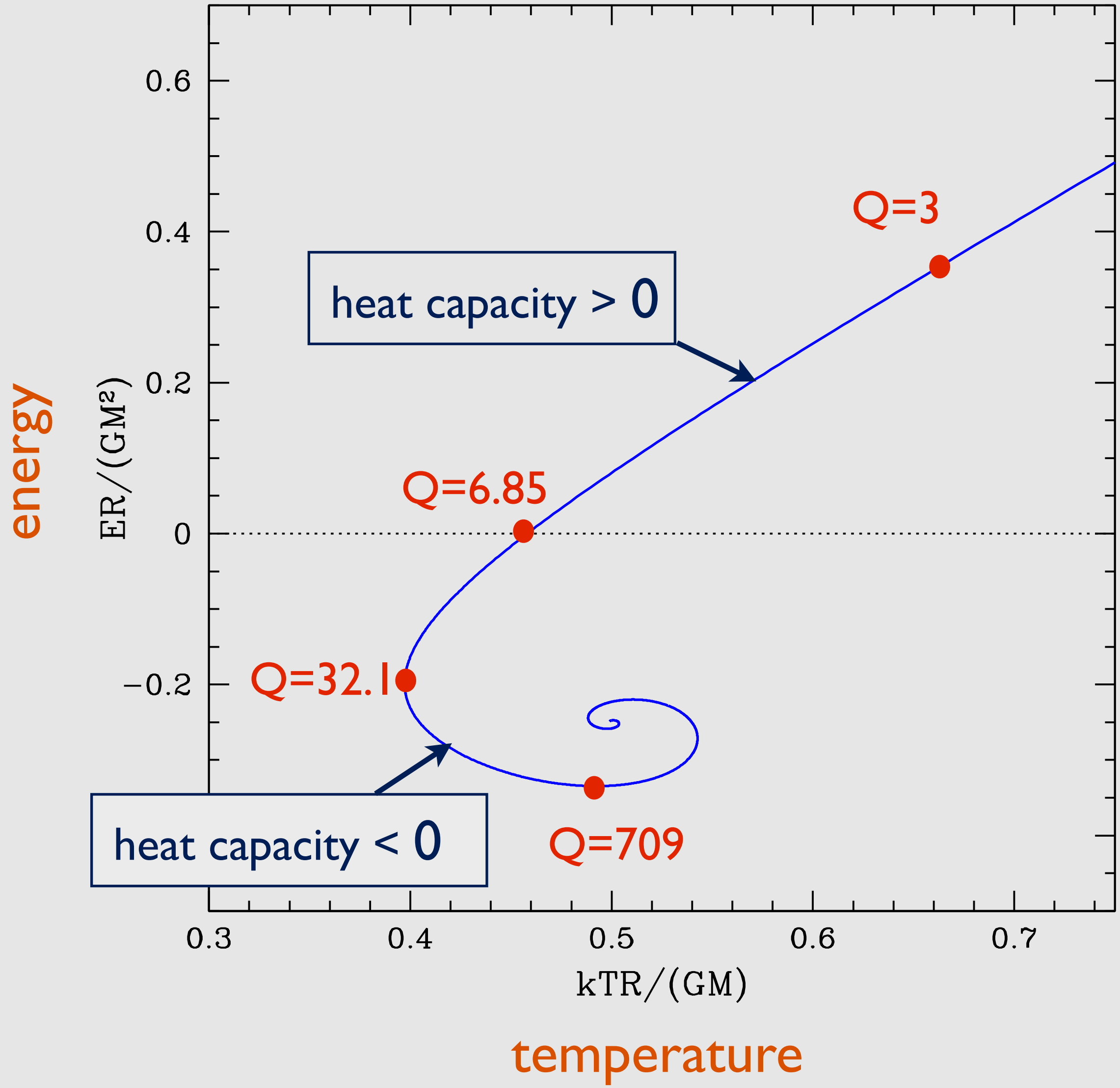
In an isothermal gas $K=3/2\ NkT$ so heat capacity is

$$C = dE/dT = -3/2\ Nk$$

which is *negative*



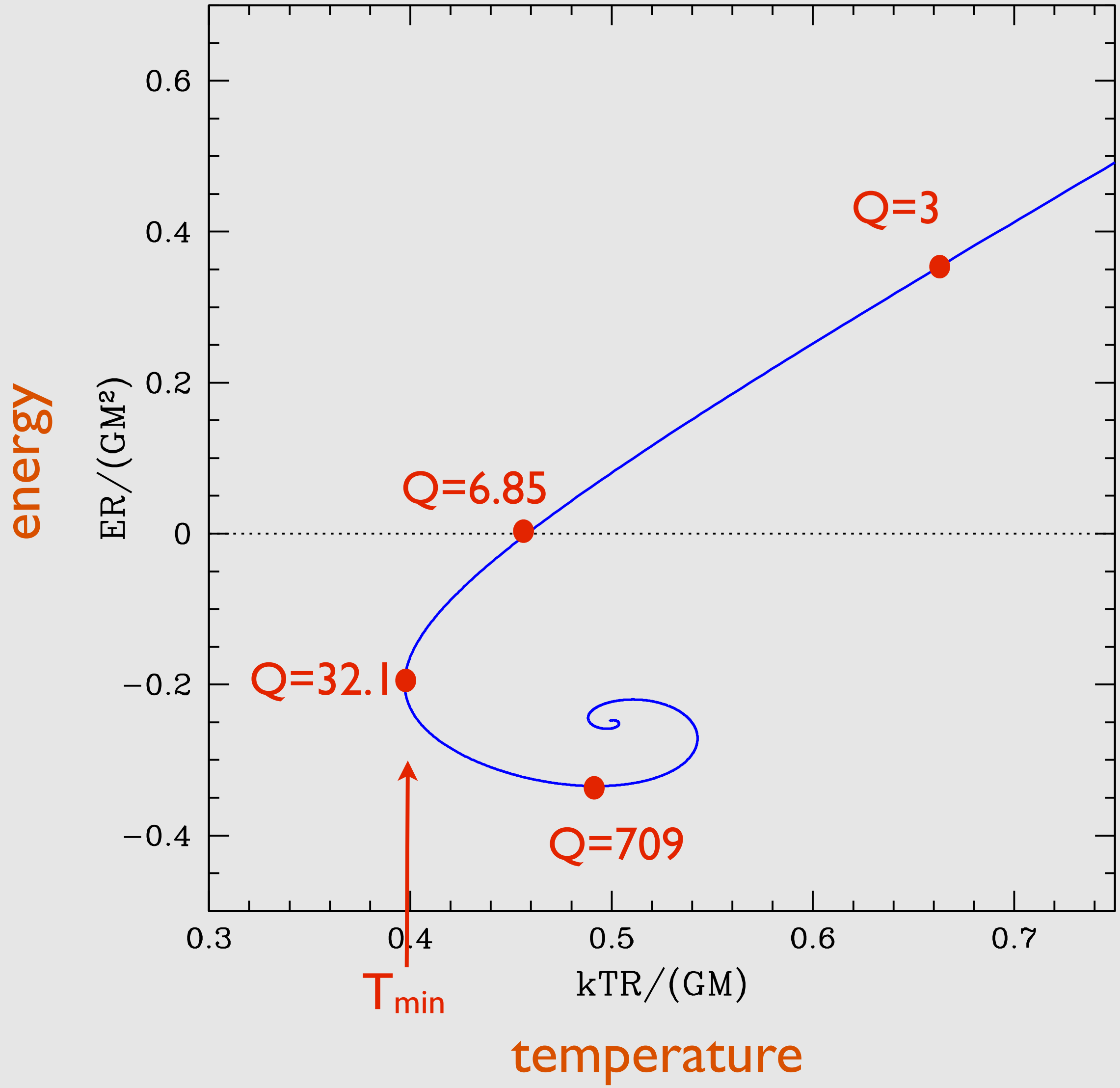
- self-gravitating gas of mass M in a rigid spherical container of radius R
- solutions parametrized by density contrast Q = $\rho(0)/\rho(R)$



- self-gravitating gas of mass M in a rigid spherical container of radius R
- solutions parametrized by density contrast Q = ρ(0)/ρ(R)
- heat capacity at constant volume C = dE/dT = slope

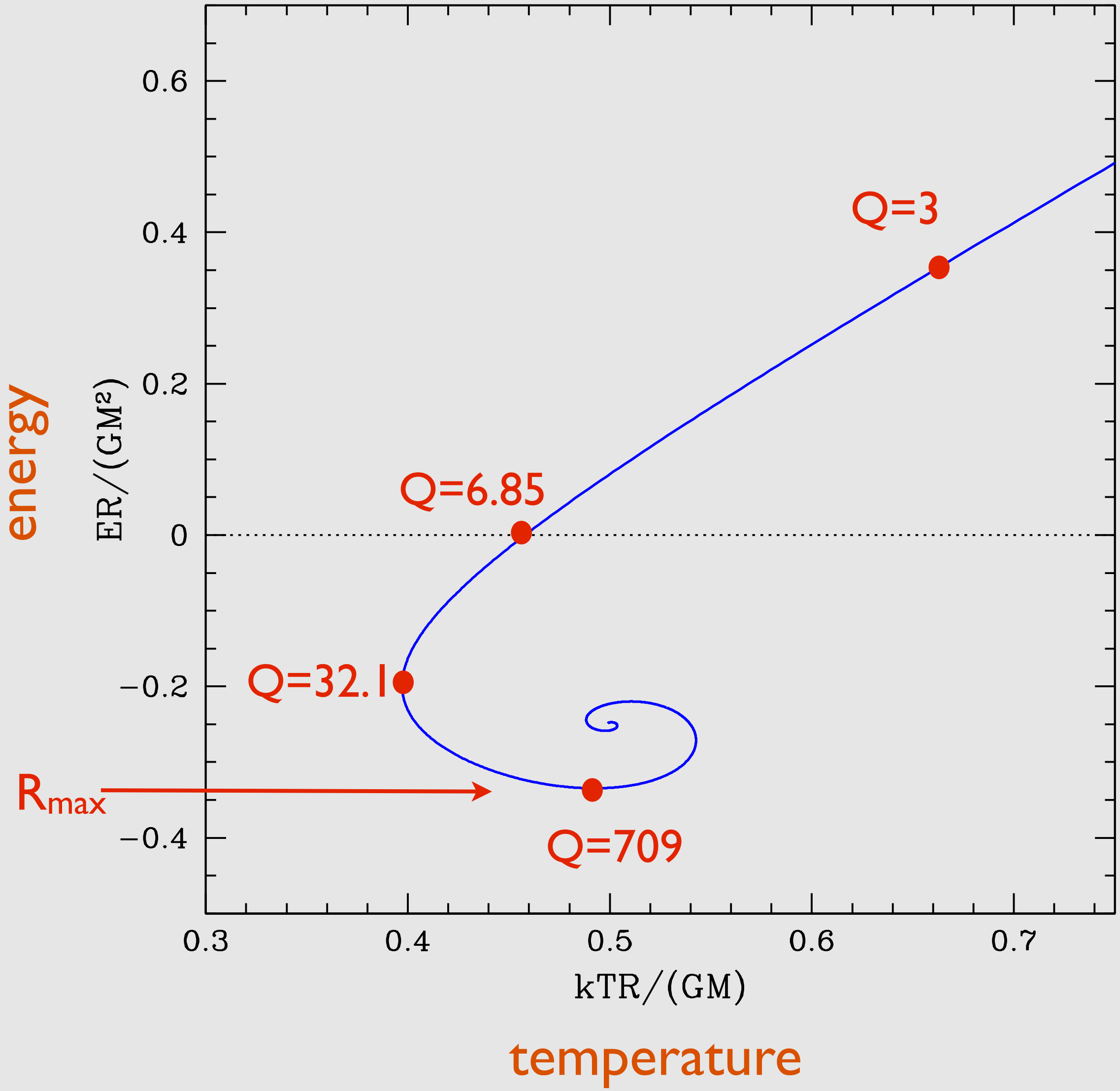
Antonov (1962)
Lynden-Bell & Wood (1968)
Thirring (1970)
Katz (1978)

thought experiment # 1:



- place box in contact with a heat bath at temperature T and slowly reduce T
- below T_{min} there is no equilibrium state

thought experiment # 2:



- insulate box and suddenly expand its radius R
- E is conserved so if E<0 $ER/(GM^2)$ becomes more negative
- for $R > R_{max}$ there is no equilibrium state
- for Q > 709 all equilibrium states are unstable (entropy is a saddle point, not a maximum)

thought experiment # 2:

- isolated self-gravitating systems have negative heat capacity
- there is no thermodynamic equilibrium state for self-gravitating systems unless they are enclosed in a sufficiently small box

lenly

<0

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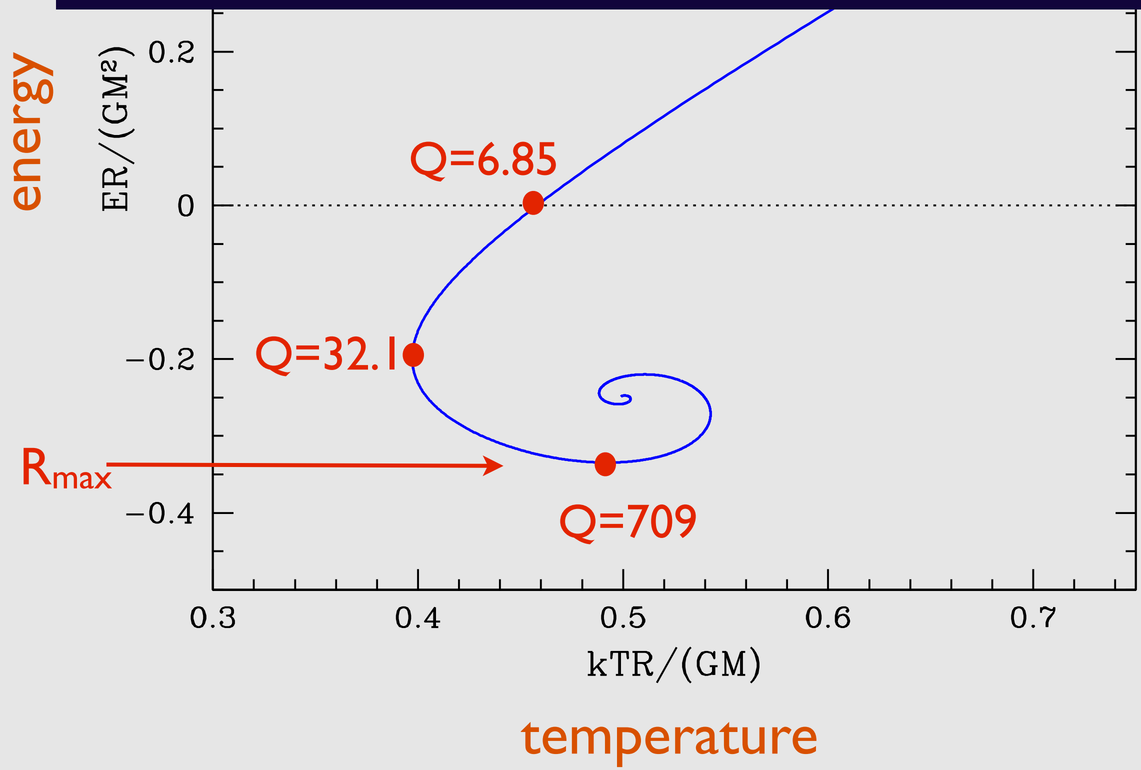
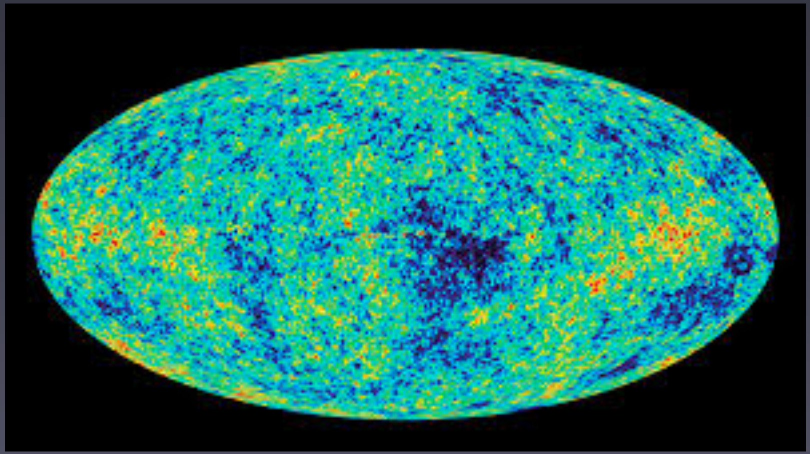


- isolated self-gravitating systems have negative heat capacity
- there is no thermodynamic equilibrium state for self-gravitating systems unless they are enclosed in a sufficiently small box
- there is no “heat death” of the Universe



- there is no thermodynamic equilibrium state for self-gravitating systems unless they are enclosed in a sufficiently small box

  ⇒ stellar systems cannot survive much longer than the equipartition or relaxation time due to gravitational encounters between stars
- for a spherical system of N stars with crossing or orbital time t_{cross}

$$t_{\rm relax} \simeq t_{\rm equipartition} \simeq 0.1\, t_{\rm cross} \frac{N}{\log N} \simeq 0.1\, \frac{R}{V} \frac{N}{\log N}$$

  where R and V are the typical size and velocity of the stellar system
- roughly, an N-body system survives for about N crossing times

globular clusters	$N \simeq 10^5$	✔
solar neighborhood	$N \simeq 10^5$	✘
dark-matter halos	$N \simeq 10^{11}$ to 10^{65}	✘
planetary systems	$N \simeq$ a few	✔ (?)
Milky Way nuclear star cluster	$N \simeq 10^5$	✔ (?)


globular clusters:
$N \simeq 10^5$
$t_{cross} \simeq 10^6$ yr
$t_{relax} \simeq 10^{10}$ yr

globular clusters:

$N \simeq 10^5$

$t_{cross} \simeq 10^6$ yr

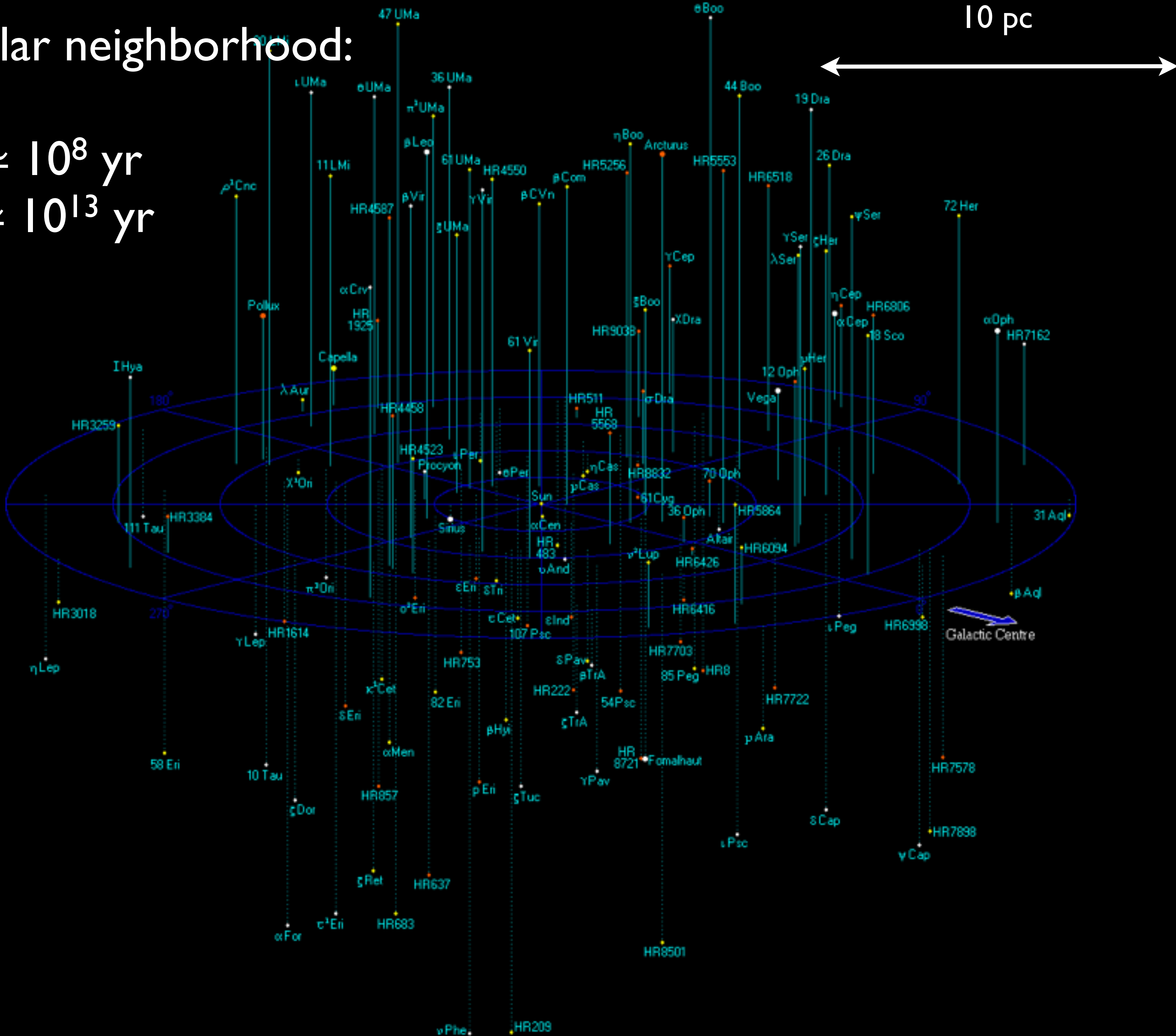
$t_{relax} \simeq 10^{10}$ yr

- core collapse
- equipartition and evaporation
- gravothermal oscillations
- Fokker-Planck approximation
- soft and hard binaries
- tidal shocks
- dynamical friction

Aarseth, Ambartsumian, Baumgardt, Cohn, Giersz, Goodman, Heggie, Hénon, Hut, Kulsrud, Larson, Lee, Lightman, Makino, Quinlan, Rasio, Shapiro, Spitzer, Spurzem, Stodółkiewicz, Sugimoto, Takahashi, etc.

the solar neighborhood:

$t_{cross} \simeq 10^8$ yr
$t_{relax} \simeq 10^{13}$ yr



- stars in the Milky Way disk exhibit velocity dispersion of 5-50 km/s in addition to common rotational velocity of ~220 km/s
- more massive stars have smaller random velocities, consistent with equipartition
- timescale required to reach equipartition due to gravitational encounters between stars is $\sim 10^{13}$ yr $\Rightarrow$ universe must be *at least* this old

TABLE I.—EQUIPARTITION OF ENERGY IN STELLAR MOTIONS.

Type of Star.	Mean Mass, M.	Mean Velocity, C.	Mean Energy, $\frac{1}{2} MC^2$.	Corresponding Temperature.
				Degrees.
Spectral type $B3$.	$19{\cdot}8 \times 10^{33}$	$14{\cdot}8 \times 10^{5}$	$1{\cdot}95 \times 10^{46}$	$1{\cdot}0 \times 10^{62}$
,, $B8{\cdot}5$.	12·9	15·8	1·62	0·8
,, $A0$.	12·1	24·5	3·63	1·8
,, $A2$.	10·0	27·2	3·72	1·8
,, $A5$.	8·0	29·9	3·55	1·7
,, $F0$.	5·0	35·9	3·24	1·6
,, $F5$.	3·1	47·9	3·55	1·7
,, $G0$.	2·0	64·6	4·07	2·0
,, $G5$.	1·5	77·6	4·57	2·2
,, $K0$.	1·4	79·4	4·27	2·1
,, $K5$.	1·2	74·1	3·39	1·7
,, $M0$.	1·2	77·6	3·55	1·7

Jeans (1928)

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- more massive stars have smaller random velocities, consistent with equipartition
- timescale required to reach equipartition due to gravitational encounters between stars is ~10^{13} yr $\Rightarrow$ universe must be *at least* this old
- in fact random velocities arise from gravitational interactions with interstellar clouds and spiral arms, and more massive stars have smaller velocities because they are younger

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elliptical galaxies:

$N \simeq 10^{11}$

$t_{cross} \simeq 10^{8}$ yr

$t_{relax} \simeq 10^{19}$ yr

elliptical galaxies:

$N \simeq 10^{11}$

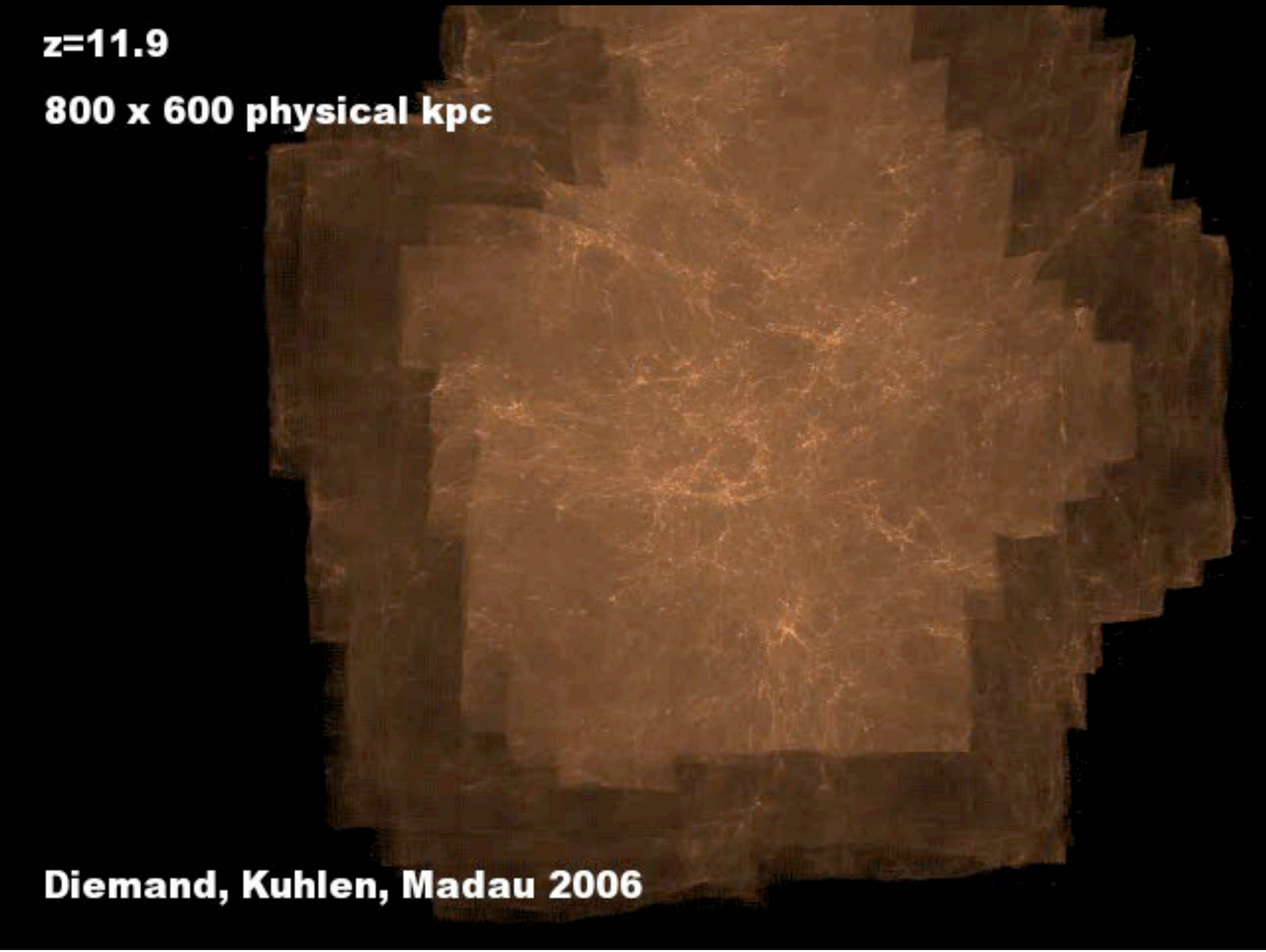
$t_{cross} \simeq 10^8$ yr

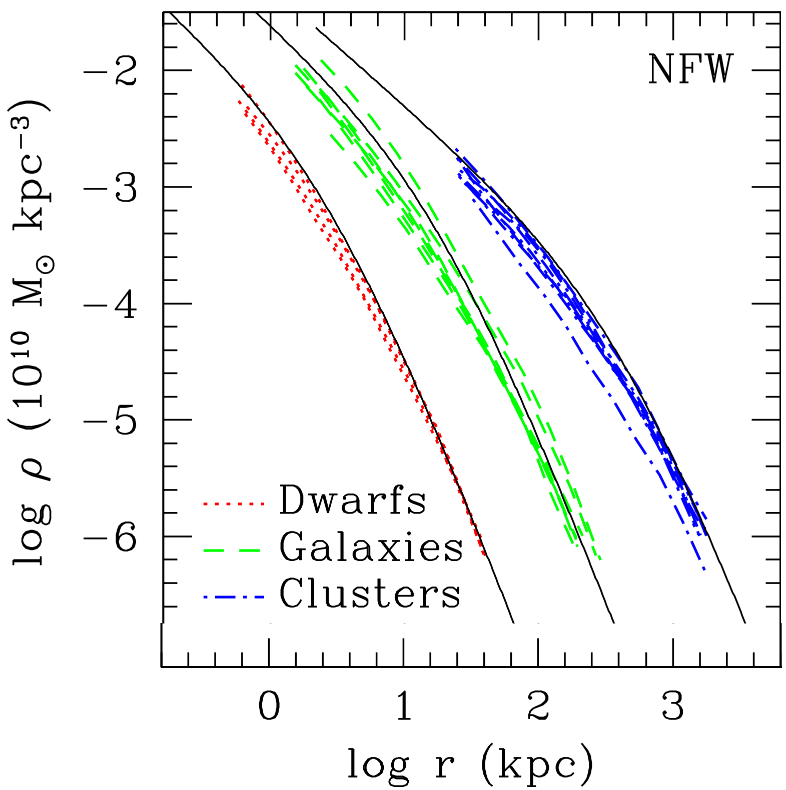
$t_{relax} \simeq 10^{19}$ yr

- the distribution of stars is similar, apart from scale, in all galaxies (Sérsic profile)
- the distribution of stellar velocities is close to Maxwellian
- how is this achieved if the relaxation time is much longer than the age? — the "fundamental paradox of stellar dynamics" (Ogorodnikov 1965)

Answer:

- large-scale fluctuations in the mean gravitational field during collapse of the galaxy drive the distribution of stars towards an (approximately) universal form ("violent relaxation", Lynden-Bell 1967)

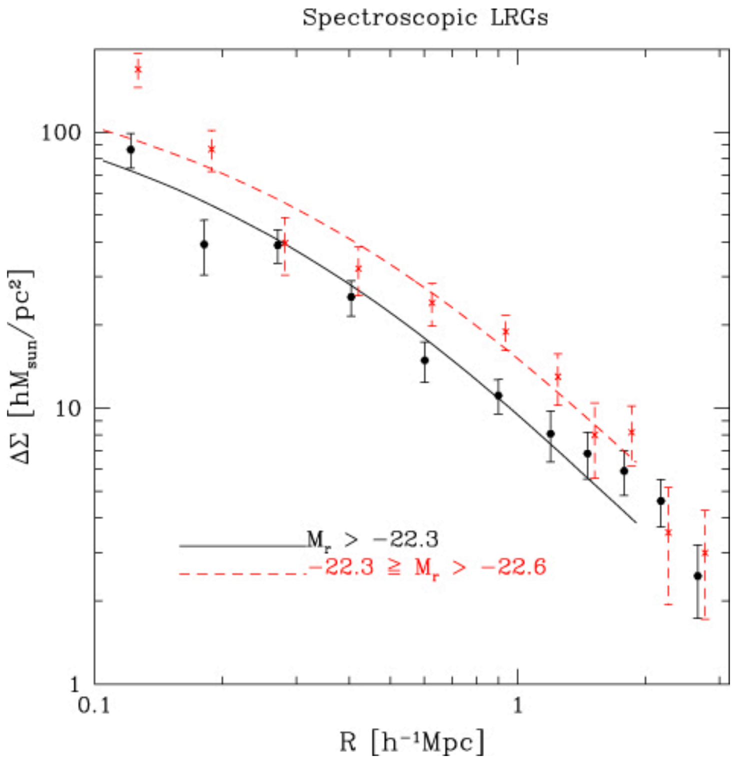

z=11.9
800 x 600 physical kpc
Diemand, Kuhlen, Madau 2006



N-body simulations,
Navarro et al. (2004)

- density profiles of dark-matter halos in simulations are well fit over > 3 orders of magnitude in radius, > 5 orders of magnitude in mass, and a wide variety of initial conditions by simple empirical formulae
- e.g., Navarro-Frenk-White (NFW) profile

$$\rho(r) = \rho_0 \frac{a^3}{r(r+a)^2}$$



weak gravitational lensing, Mandelbaum et al. (2008)

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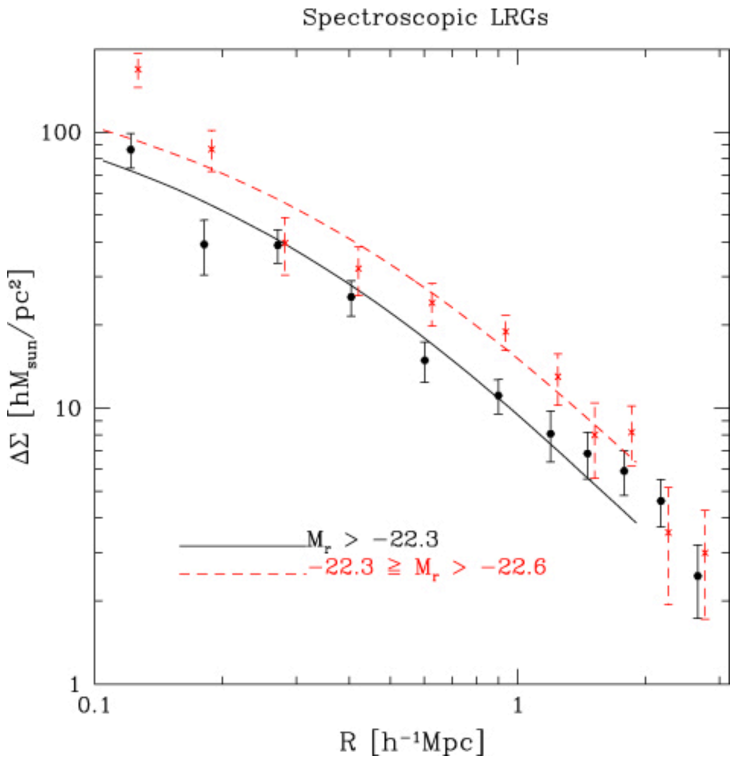

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$$\rho(r) = \rho_0 \frac{a^3}{r(r+a)^2}$$

- suggests that there is some simple physics that determines the density profile and other halo properties

Lynden-Bell (1967)
Saslaw (1968, 1969,1970)
Shu (1969,1978,1987)
Lecar & Cohen (1971)
Miller (1972)
Severne & Luwel (1980)
Binney (1982)
Rephaeli (1983)
Madsen (1987)
Stiavelli & Bertin (1987)
White & Narayan (1987)
Kandrup (1989)
Soker (1990,1996)
Hjorth & Madsen (1991,1993)
Chavanis (1998,2002,2006)
Ascasibar et al. (2004)
Arad et al. (2004)
Hansen et al. (2005,2010)
Arad & Lynden-Bell (2005)
Lu et al. (2006)
Valluri et al. (2007)
Williams & Hjorth (2010)
Dalal et al. (2010)
Visbal et al. (2012)
Pontzen & Governato (2013)
Beraldo e Silva et al. (2014)
Alard (2014)

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In most dark matter models the phase-space density $f(\mathbf{x}, \mathbf{v})$ satisfies the collisionless Boltzmann equation (a.k.a. Vlasov equation, Liouville equation, continuity equation in phase space)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

and the Poisson equation

$$\nabla^2 \Phi = 4\pi G \int d\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t).$$

The natural first approach is to assume that violent relaxation leads to a final state that maximizes the entropy

$$S = -\int d\mathbf{x} d\mathbf{v}\, f \log f + \text{constant}$$

Boltzmann entropy

at fixed mass and energy.

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Boltzmann entropy

at fixed mass and energy.

The primary feature of entropy in statistical mechanics is that it satisfies Boltzmann's H theorem, i.e. molecular collisions imply that

$$\frac{dH}{dt} \leq 0 \quad \text{where} \quad H = -S = \int d\mathbf{x} d\mathbf{v} \, f \log f.$$

Boltzmann entropy

This calculation is based on several strong assumptions:

- instantaneous binary collisions
- short-range forces
- molecular chaos

What kind of H-theorem applies to violent relaxation?

Most general approach is to treat relaxation is a Markov process in phase space defined by the probability p_{ji} that a particle in cell i at the initial time transitions to cell j at the final time. If all cells have the same size then time-reversibility implies $p_{ji} = p_{ij}$.

Then one can show

$$\frac{dH}{dt} \leq 0 \quad \text{where} \quad H = \int d\mathbf{x}d\mathbf{v}C(f)$$

and $C(f)$ is any convex function, $C''(f) \geq 0$, e.g.,

$$C(f) = f\log f, \quad C(f) = f^2, \quad C(f) = -\log f, \text{ etc.}$$

Boltzmann entropy

In this process the Boltzmann entropy *has no special status.* There is a different "entropy" for every convex function and all must go up.

An initial phase-space distribution f(x,v) can only evolve into a final one f′(x,v) if *all* possible H-functions are smaller for f′ than for f.

Simpler criteria exist (Tremaine, Hénon & Lynden-Bell 1987, Yu & Tremaine 2002, Dehnen 2005)

$$\int d\mathbf{x}d\mathbf{v}\,\max[f(\mathbf{x},\mathbf{v})-\phi,0] \geq \int d\mathbf{x}d\mathbf{v}\,\max[f'(\mathbf{x},\mathbf{v})-\phi,0] \quad \text{for all } \phi > 0$$

In classical statistical mechanics, relaxation leads to a unique equilibrium distribution function f(x,v)

Violent relaxation leads to a partial ordering of distribution functions f(x,v)

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Dehnen (2005)

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Violent relaxation leads to a partial ordering of distribution functions f(x,v)

Unfortunately for cold dark matter the left side diverges...
⇒ some physics other than maximum entropy is needed to understand violent relaxation

Statistical mechanics of planetary systems

There are many bad examples of attempts to explain the spacing and other properties of planetary orbits from first principles 

Nevertheless there are reasons to try again:

- *Kepler* has provided a large statistical sample of multi-planet systems
- N-body integrations can routinely follow the evolution of systems for 100 Myr
- there are hints of interesting behavior from studies of the solar system:
  - the orbits of the planets in the solar system are chaotic, with Liapunov (e-folding) times of $\sim 10^7$ yr (Sussman & Wisdom 1988, 1992, Laskar 1989, Hayes 2008)
  - the outer solar system is "full" in the sense that no stable orbits remain between Jupiter and Neptune (Holman 1997)
  - there is a 1% chance that Mercury will be lost from the solar system before the end of the Sun's life in ~ 7 Gyr

These suggest that some properties of the solar system might be determined by the statistical mechanics of orbital chaos

Statistical mechanics of planetary systems

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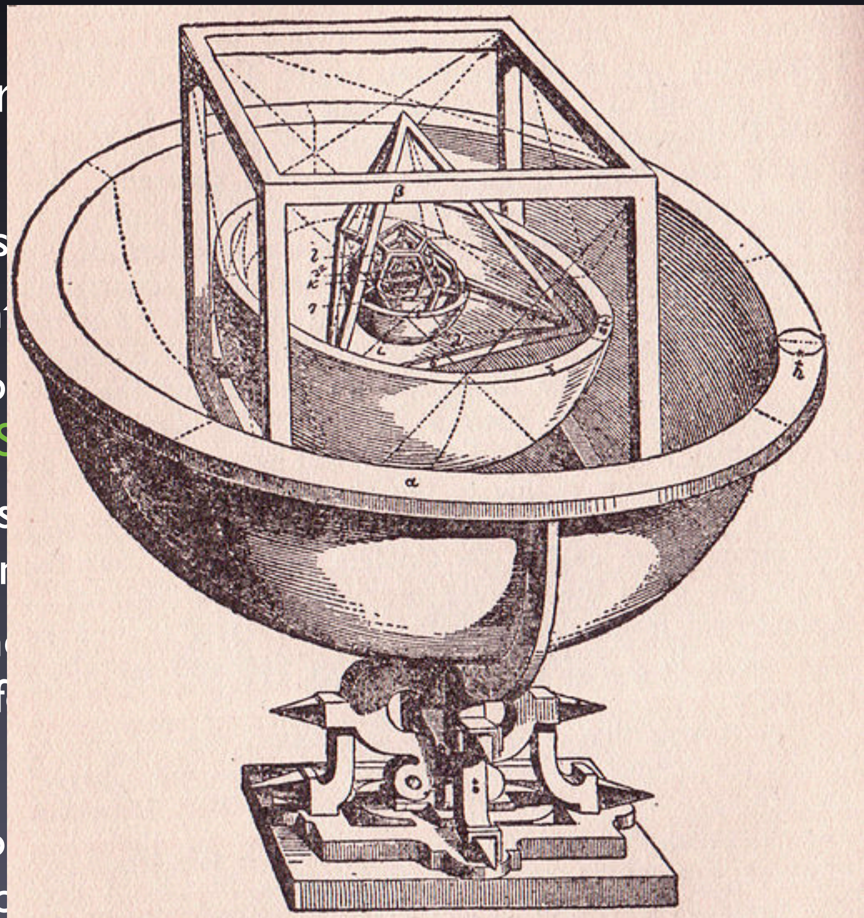

Fig. 37 —Kepler's Analogy of the Five Solids and the Five Worlds.

Nevertheless ther

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- N-body integrations ... ns for 100 Myr
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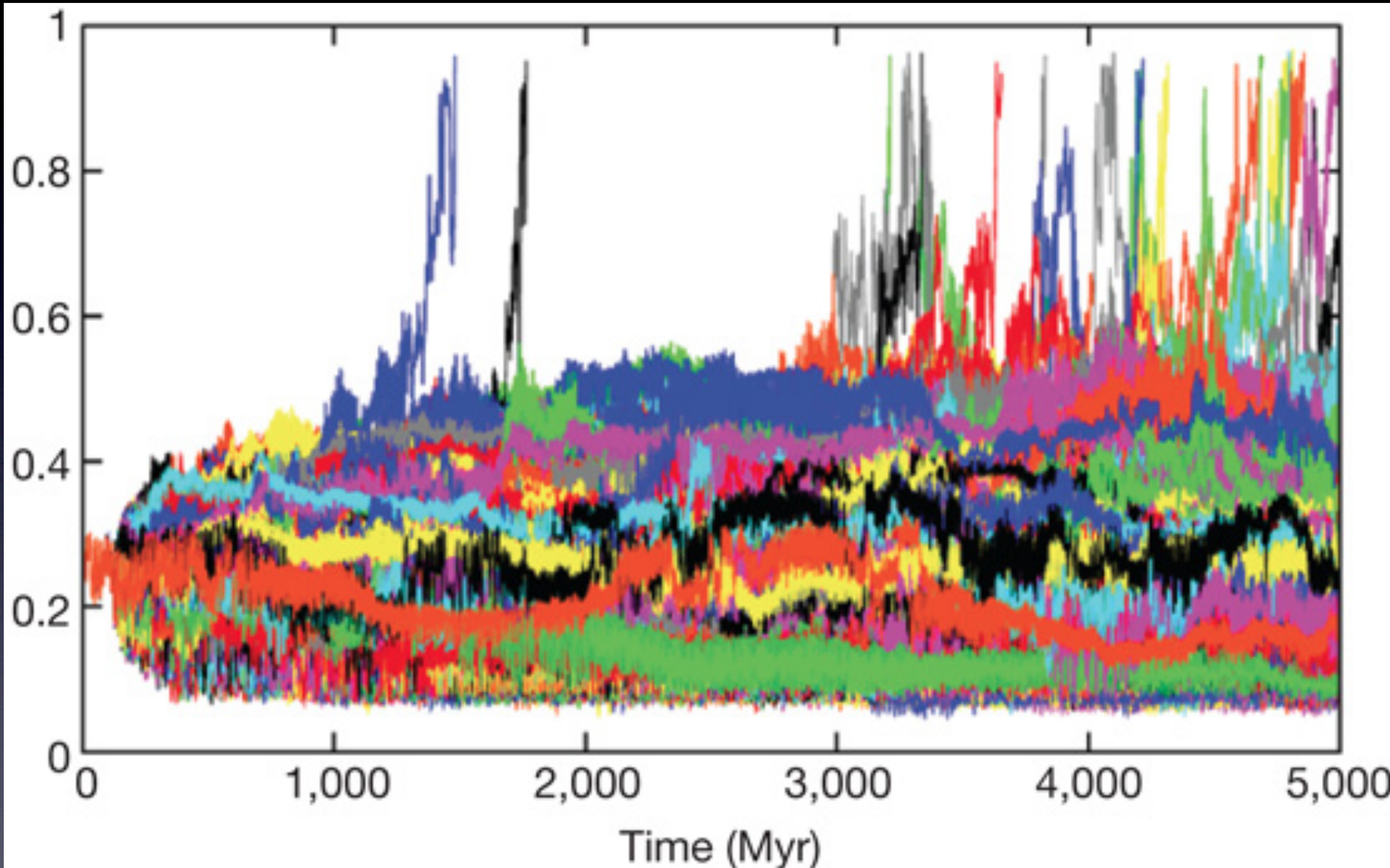
These suggest that some properties of the solar system might be determined by the statistical mechanics of orbital chaos eccentricity of Mercury for 2500 nearby initial conditions



Laskar & Gastineau (2009)

The last stages of terrestrial planet formation

- the accretion of planetesimals leads to a few dozen "planetary embryos" of similar size
- eccentricities of the embryos remain small because they are damped by dynamical friction from residual population of small planetesimals
- eventually the small planetesimals disappear
- surviving embryos gradually excite one another's eccentricities until their orbits cross and they collide
- through collisions, the number of surviving bodies slowly declines until we are left with a small number of planets on well-separated, stable orbits (late-stage accretion, post-oligarchic growth, giant-impact phase)
- maybe the giant-impact phase tends to produce an ensemble of planetary systems with statistically similar properties

Maybe the giant-impact phase tends to produce an ensemble of planetary systems with statistically similar properties. This idea is not new:

- Laskar (1996): "maybe there was some extra planet at the early stage of formation of the solar system...but this led to so much instability that one of the planets...suffered a close encounter or a collision. This leads eventually to the escape of the planet and the remaining system gets more stable...at each stage, the system should have a time of stability comparable with its age."
- Barnes & Raymond (2004): proposed the *packed planetary systems hypothesis*: "many systems lie on the verge of instability. Planetary systems near instability are as tightly packed as possible; there is no room for additional companions."
- Malhotra (2015): "as a consequence of mergers or ejections of planets, the surviving planets undergo a random walk of their orbits; unstable configurations are steadily winnowed."
- Volk & Gladman (2015): "systems of tightly packed inner planets initially surrounded nearly all such stars and those observed are the final survivors of a process in which long-term metastability eventually ceases and the systems proceed to collisional consolidation or destruction"
- Pu & Wu (2015): "we suggest that typical planetary systems were formed with even tighter spacing, but most, except for the widest ones, have undergone dynamical instability, and are pared down to a more anemic version of their former selves, with fewer planets and larger spacings."

Statistical mechanics of planetary systems

The range of strong interactions from a planet of mass m orbiting a star of mass M in a circular orbit of radius a is the Hill radius

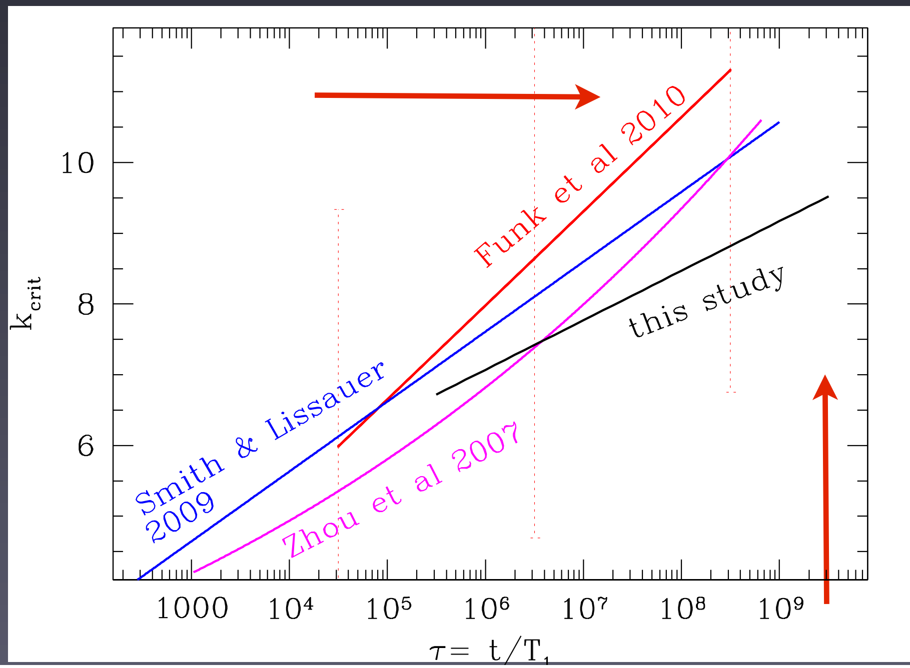
$$r_H = a\left(\frac{m}{3M}\right)^{1/3}.$$

Numerical integrations show that planets of mass m, m′ with semi-major axes a, a', a < a' are stable for N orbital periods if closest approach exceeds k Hill radii, or

$$a'(1-e') - a(1+e) > k(N) r_H$$

$$r_H = \bar{a}\left(\frac{m+m'}{3M}\right)^{1/3}.$$

typically $k(10^{10}) \simeq 11 \pm 1$



Pu & Wu (2014)

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$$\left(m \right)^{1/3}$$

Ansatz: planetary systems fill uniformly the region of phase space allowed by stability (~ ergodic hypothesis)

Nu … a’,
a < a’ are stable for N orbital periods if closest approach exceeds k Hill radii, or

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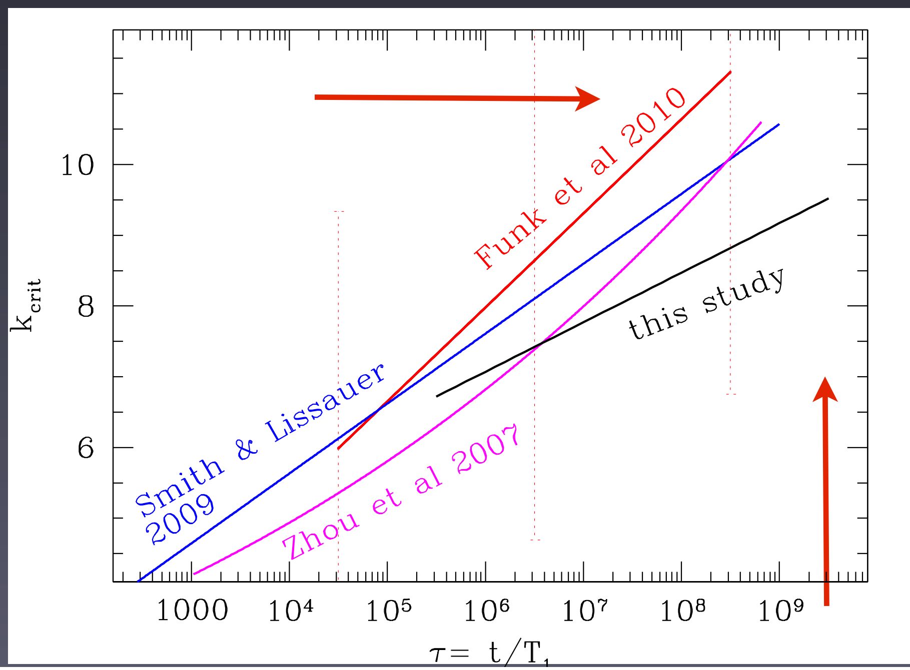


Statistical mechanics of planetary systems

1. Ansatz: planetary systems fill uniformly the region of phase space allowed by stability

2. Work in the sheared sheet model, which replaces usual Keplerian disk by a rectangular box with shear (not essential, but eliminates spatial gradients and other complications)

Leads to an N-planet distribution function

phase-space volume

apocenter and pericenter must be separated by k Hill radii

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} \overbrace{da_i de_i^2} H[a_{i+1}(1-e_{i+1}) - a_i(1+e_i) - kr_H]$$

step function

where $H(\cdot)$ is the step function, $k = 11 \pm 1$, and $r_H = \bar{a}(m_i + m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

For comparison the distribution function for a one-dimensional gas of hard rods of length L (Tonks 1936) is

$$p(a_1, \ldots, a_N) \propto \prod_{i=1}^{N} da_i H(a_{i+1} - a_i - L).$$

In both systems the partition function depends only on the filling factor

$$F = \frac{k\langle r_H \rangle}{\langle a_{i+1} - a_i \rangle}, \qquad F = \frac{L}{\langle a_{i+1} - a_i \rangle}.$$

Statistical mechanics of planetary systems

N-planet distribution function

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} da_i de_i^2 \, H[a_{i+1}(1 - e_{i+1}) - a_i(1 + e_i) - kr_H]$$

where $H(\cdot)$ is the step function, $k = 11 \pm 1$, and $r_H = \bar{a}(m_i + m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

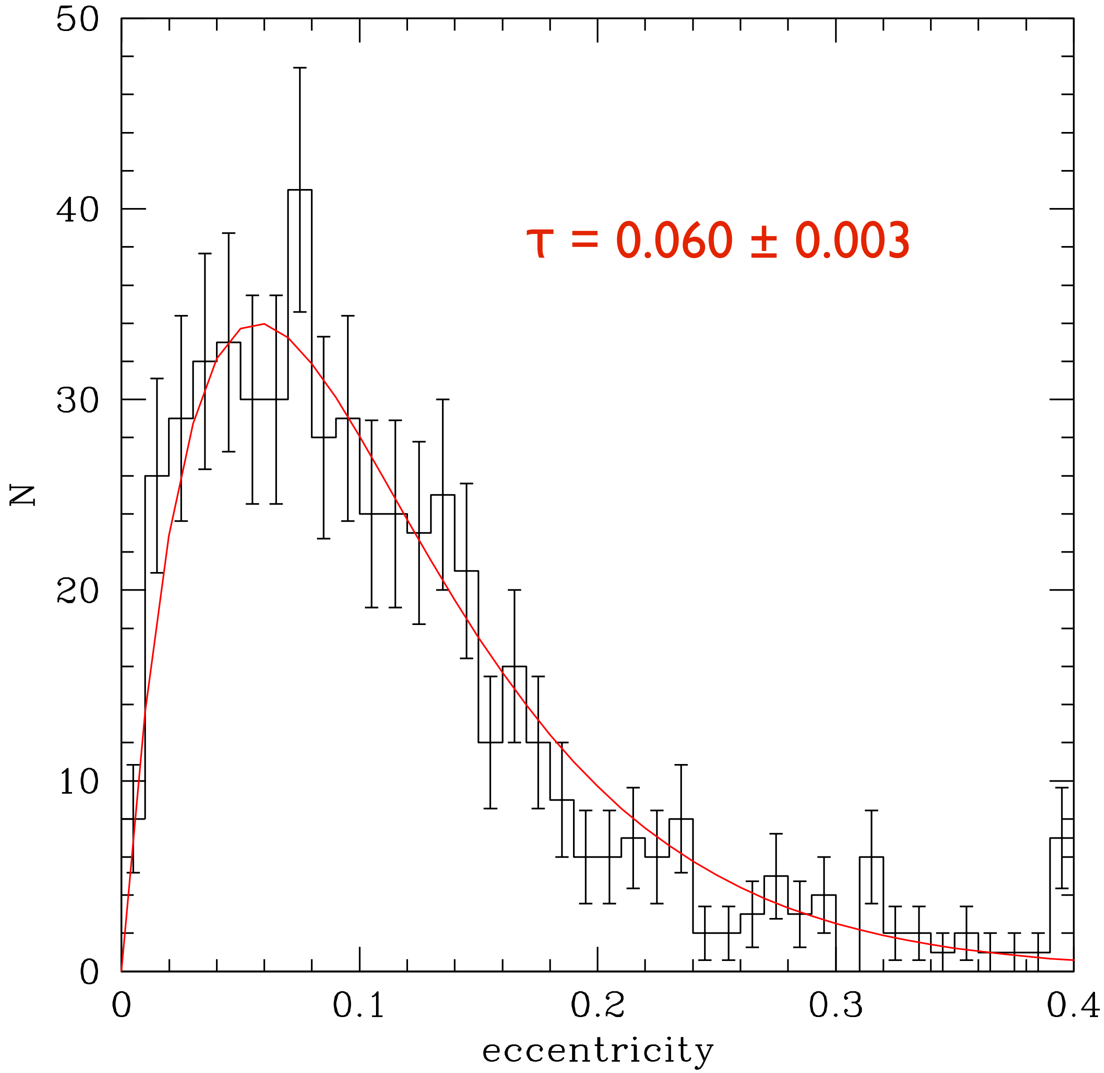
Work with the grand canonical ensemble, i.e., assume each planetary system is a subsystem with variable number of planets

Predictions:

- eccentricity distribution:

$$p(e) = \frac{e}{\tau^2} \exp\left(-\frac{e}{\tau}\right)$$

where τ is a free parameter determined by the filling factor e.g., N-body simulations of planet growth by Hansen & Murray (2013)



Statistical mechanics of planetary systems

N-planet distribution function

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} da_i de_i^2 \, H[a_{i+1} - a_i - \bar{a}(e_{i+1} + e_i) - kr_H]$$

where $H(\cdot)$ is the step function, $k = 11 \pm 1$, and $r_H = \bar{a}(m_i + m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

Work with the grand canonical ensemble.

Predictions:

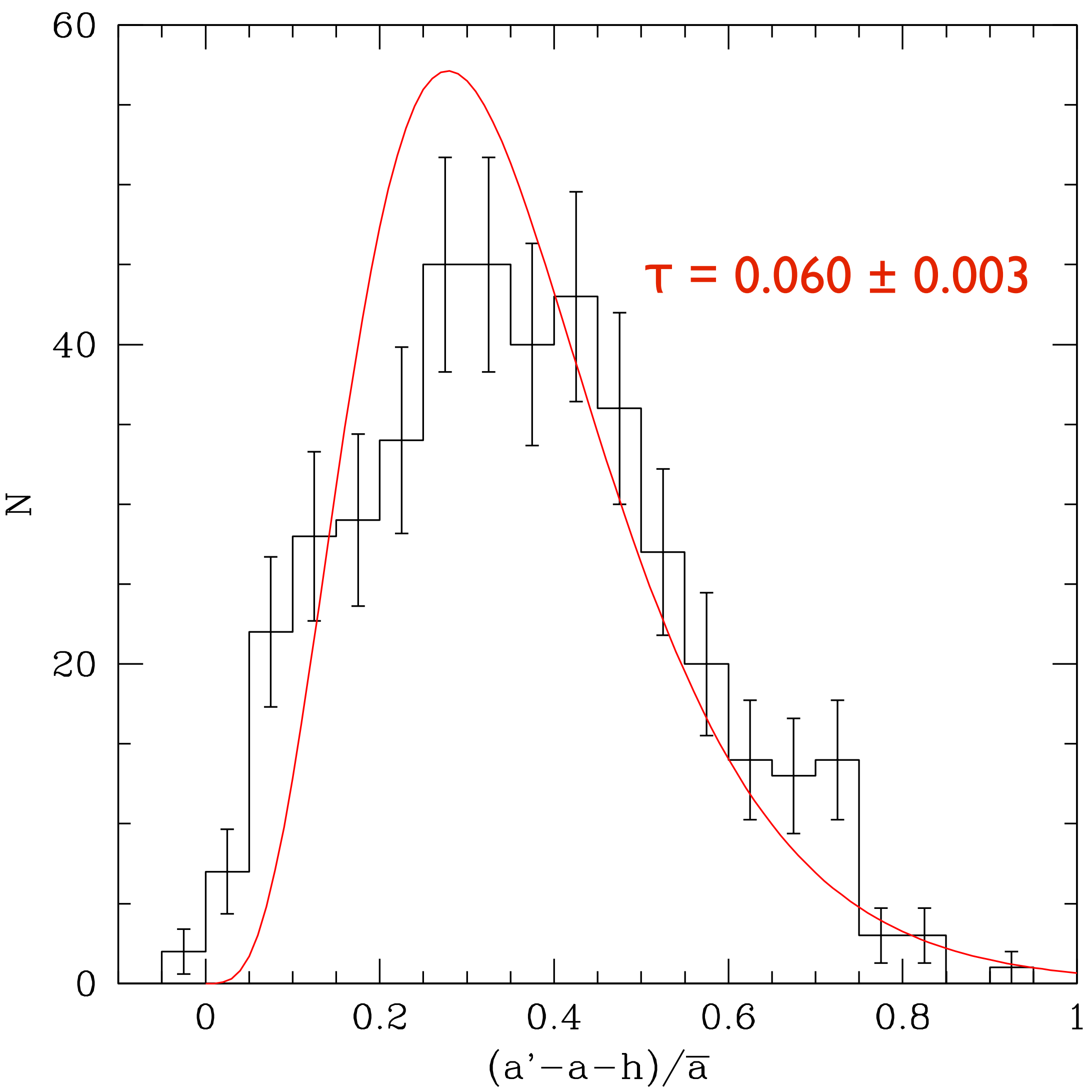
- eccentricity distribution ✔ with one free parameter
- distribution of semi-major axis differences between nearest neighbors:

$$p(a' - a) = \frac{4}{2\bar{a}\tau} G\left(\frac{a' - a - kr_H}{2\bar{a}\tau}\right)$$

where

$$G(x) = 6\exp(-x) - \exp(-2x)(x^3 + 3x^2 + 6x + 6).$$

e.g., N-body simulations of planet growth by Hansen & Murray (2013)



Statistical mechanics of planetary systems

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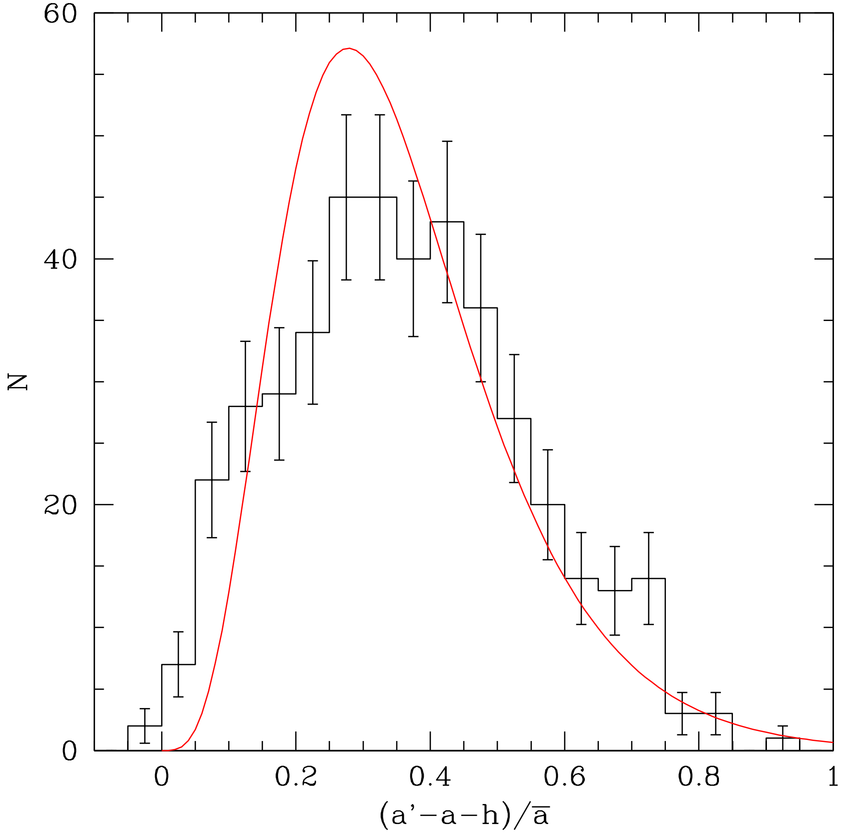
Work with the grand canonical ensemble.

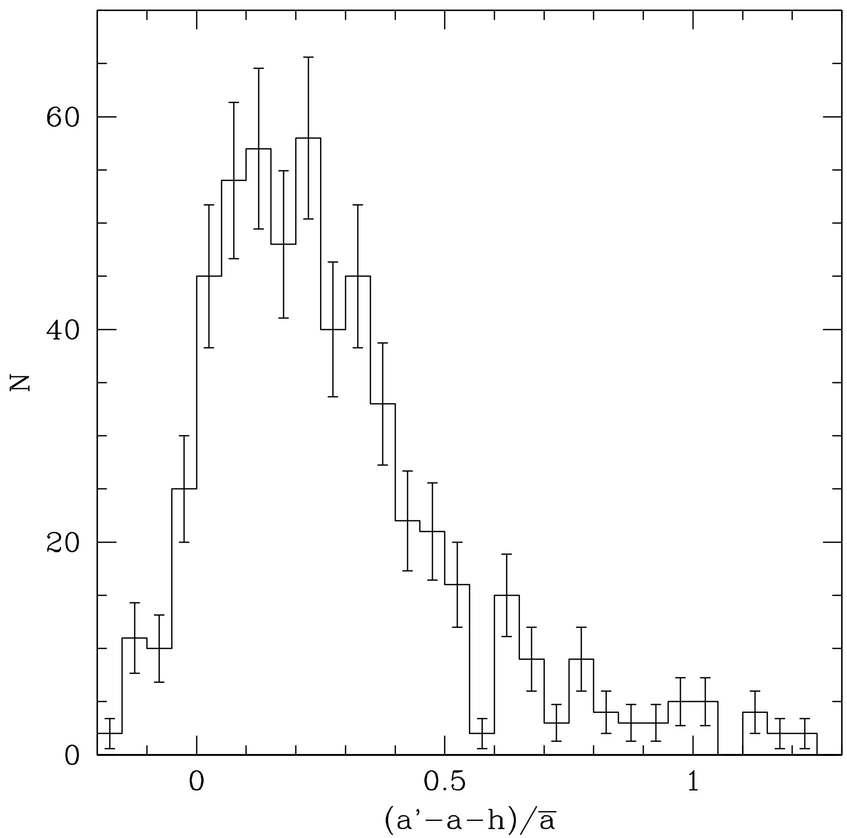
Predictions:

- eccentricity distribution ✔ with one free parameter
- distribution of semi-major axis differences ✔ with no free parameters

Hansen & Murray (2013) simulations



Kepler planets, using Weiss & Marcy (2014) mass-radius relation:



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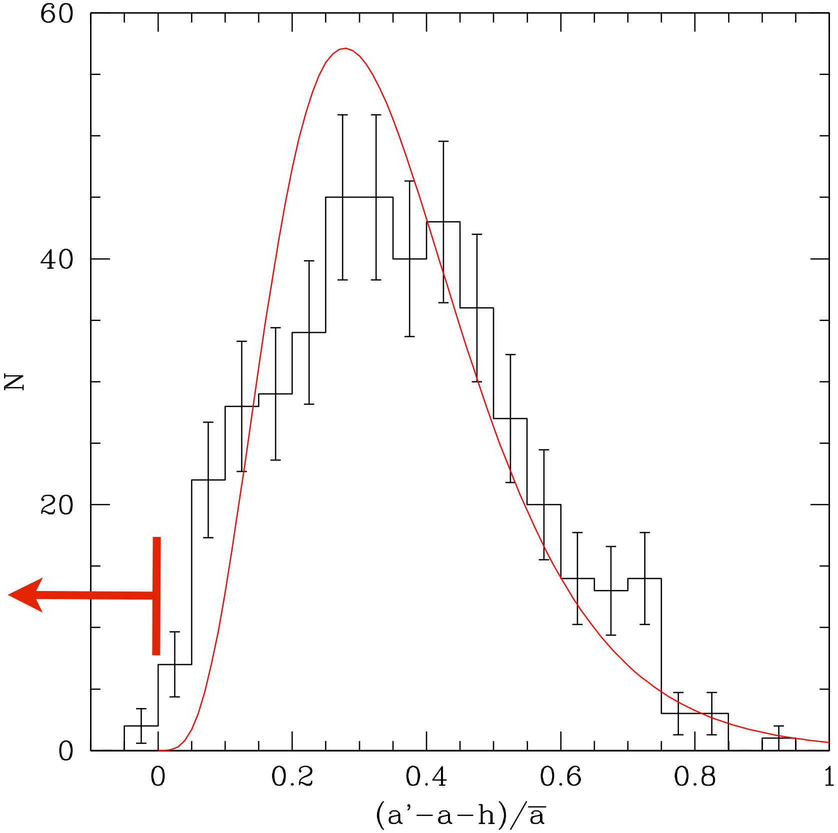

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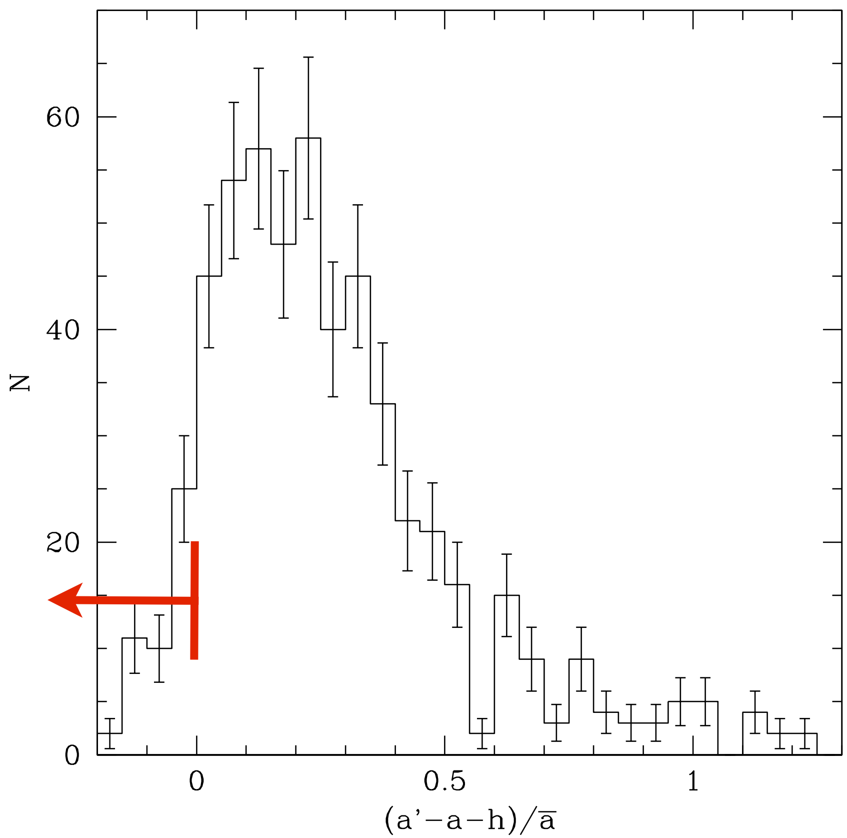

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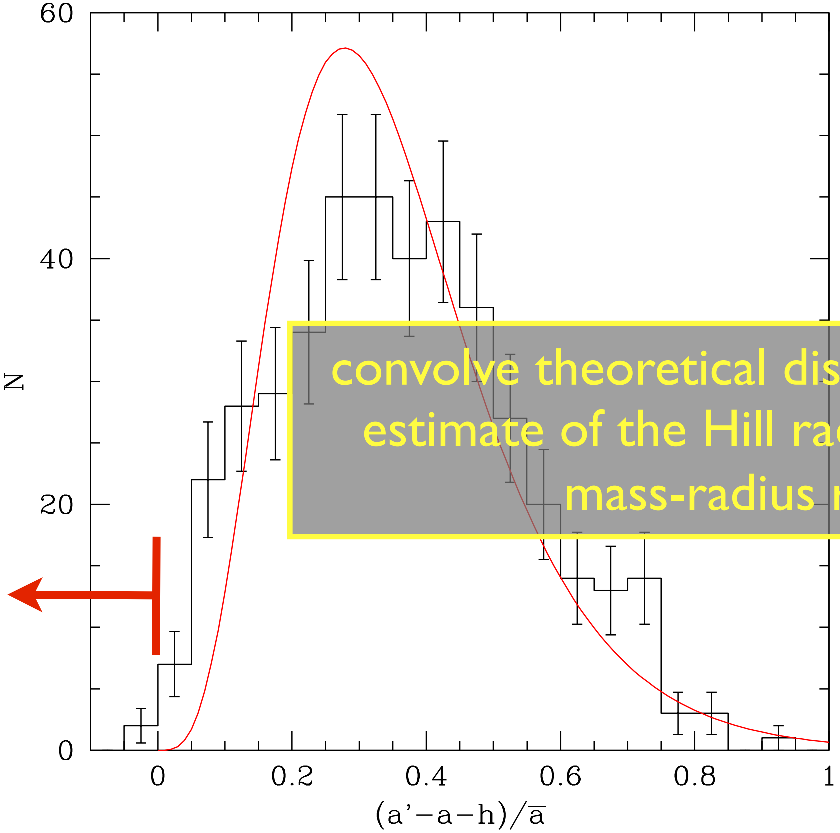
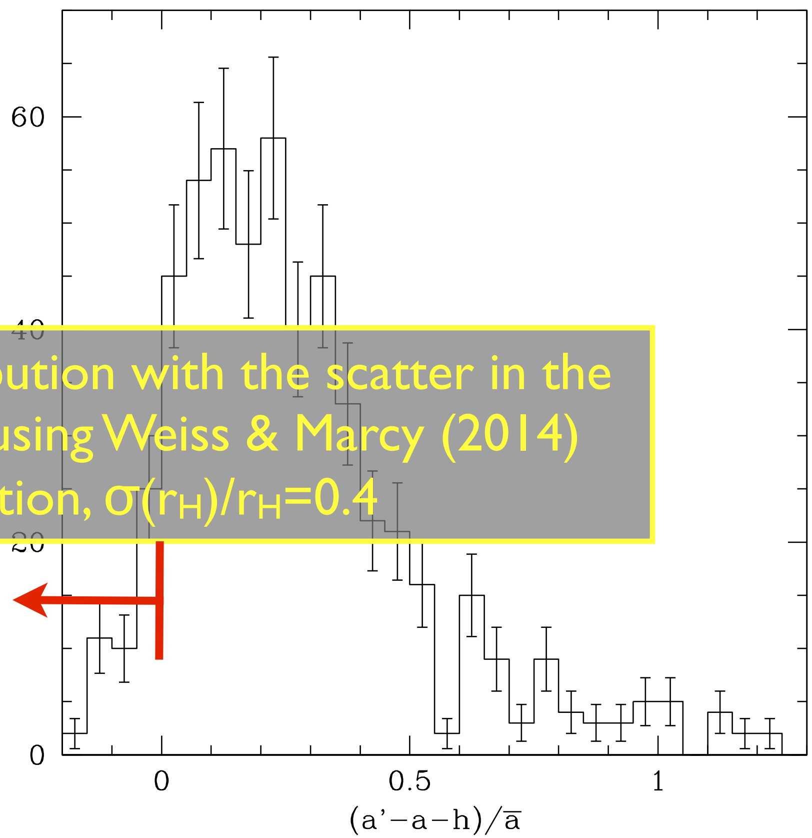
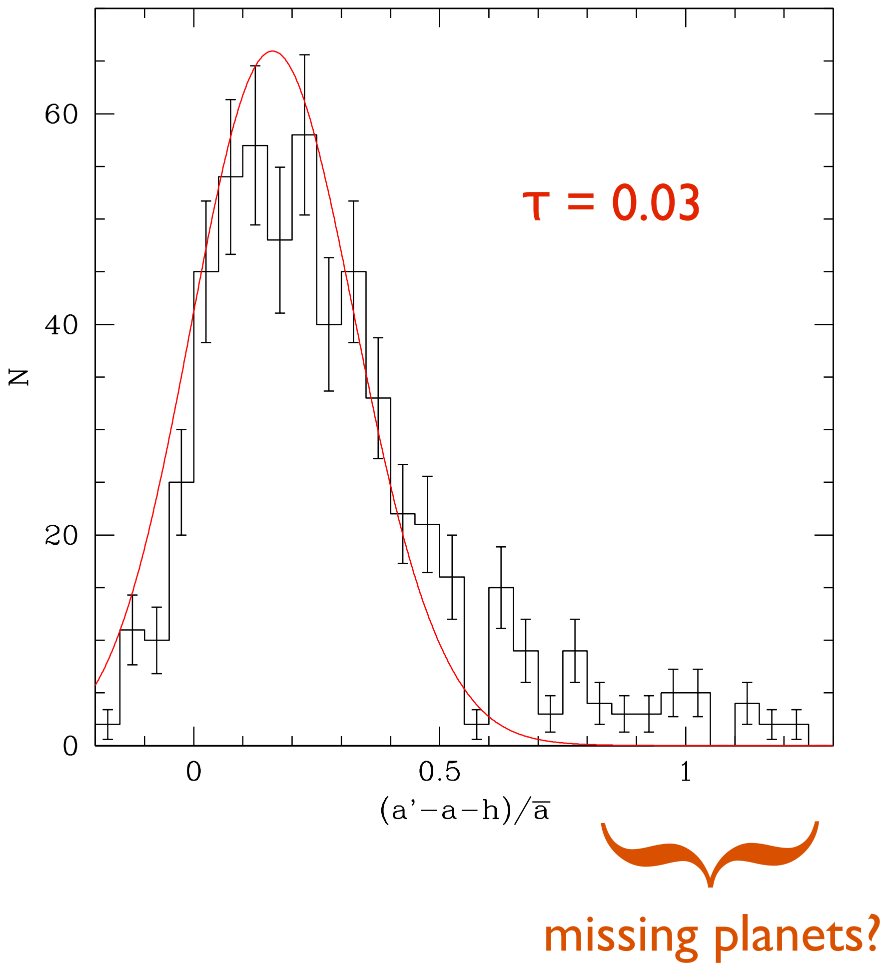




convolve theoretical distribution with the scatter in the estimate of the Hill radii using Weiss & Marcy (2014) mass-radius relation, $\sigma(r_H)/r_H=0.4$

Kepler planets, using Weiss & Marcy (2014) mass-radius relation:



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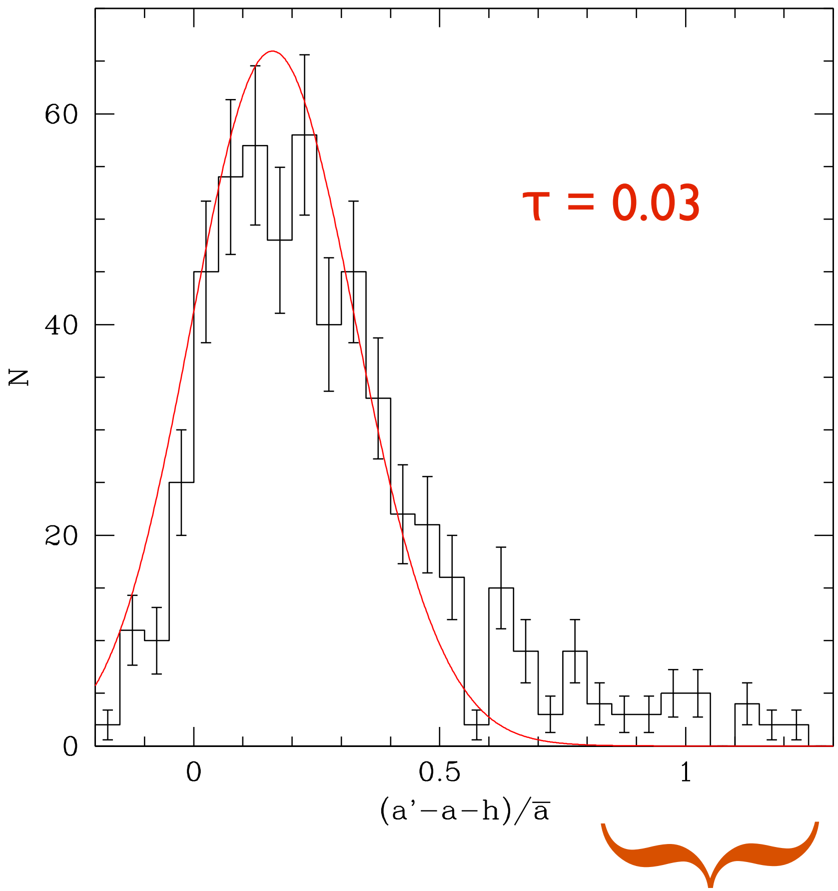

missing planets?

- with τ=0.03 ergodic model predicts <e>=0.06
  - <e>≃0.02-0.03 (Hadden & Lithwick 2014,2015)
  - <e>≃0.03 (Fabrycky et al. 2014)

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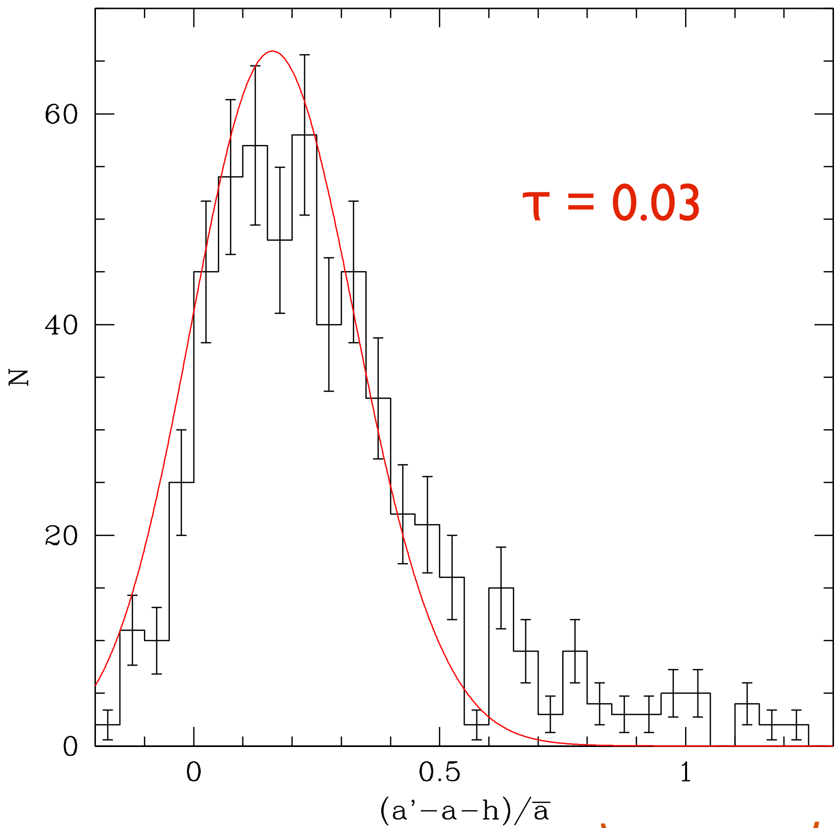
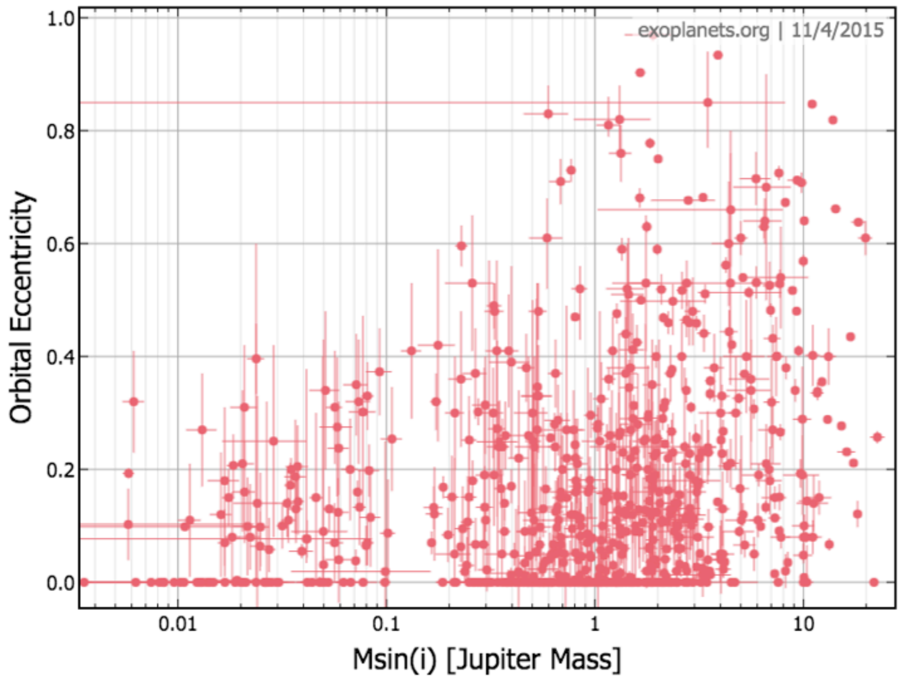
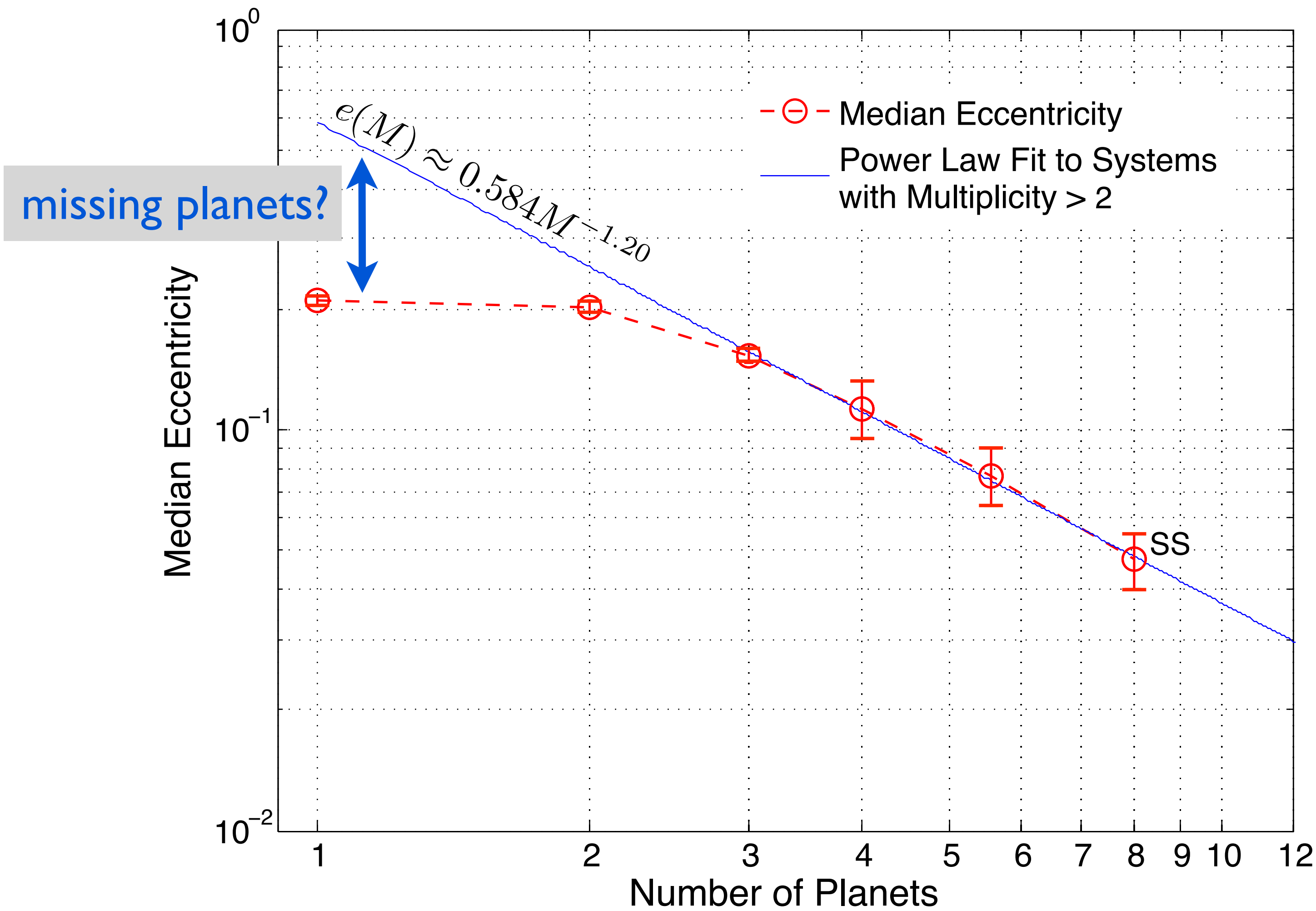


missing planets?

- with τ=0.03 ergodic model predicts <e>=0.06
  - <e>≃0.02-0.03 (Hadden & Lithwick 2014,2015)
  - <e>≃0.03 (Fabrycky et al. 2014)
- ergodic model predicts no correlation between mass and eccentricity in a given system





Limbach & Turner (2014)

- ergodic model predicts <e> ~ 1/N

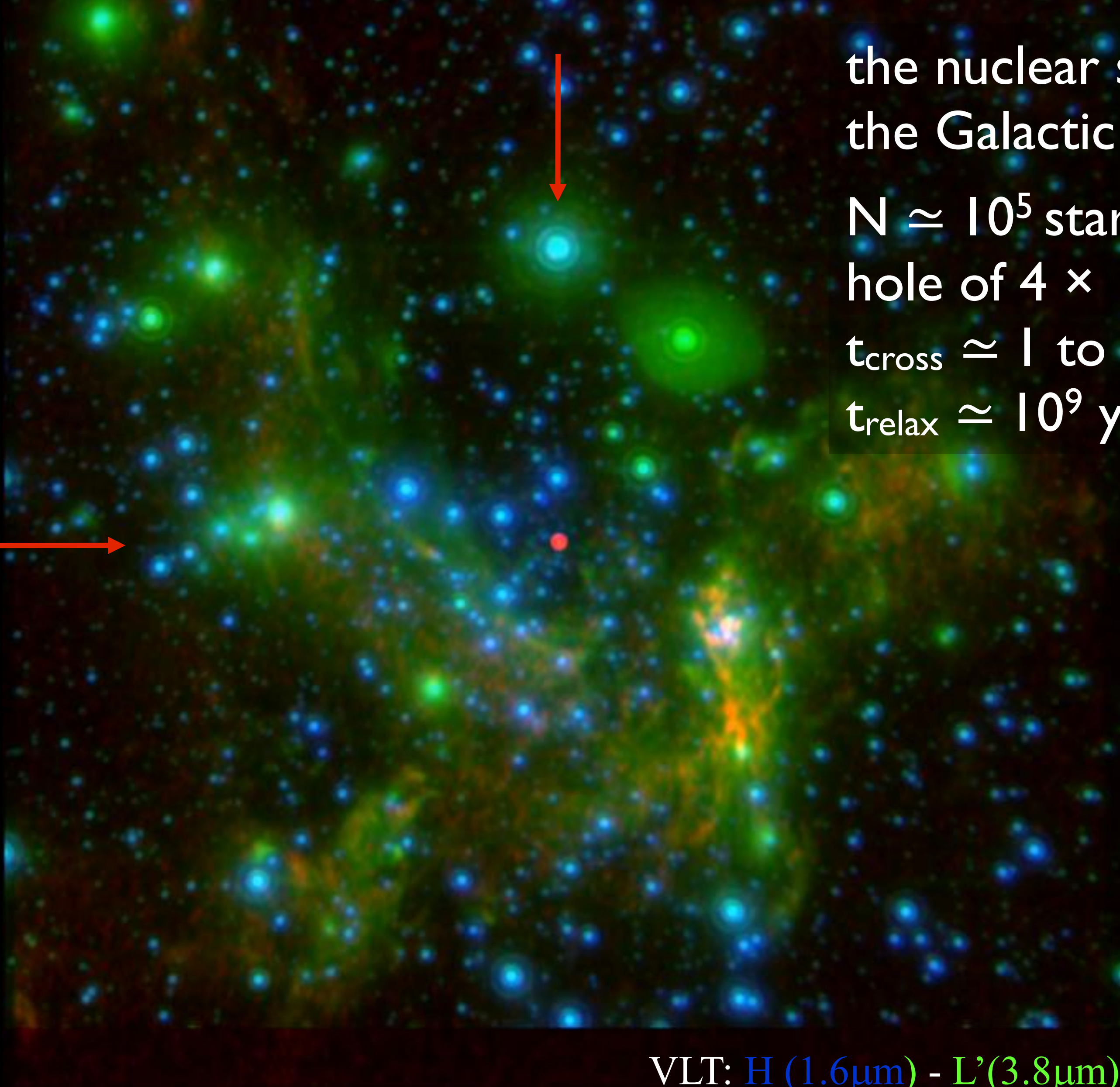
the nuclear star cluster at the Galactic center:

$N \simeq 10^5$ stars plus a black hole of $4 \times 10^6\ M_\odot$

$t_{cross} \simeq 1$ to 10^4 yr

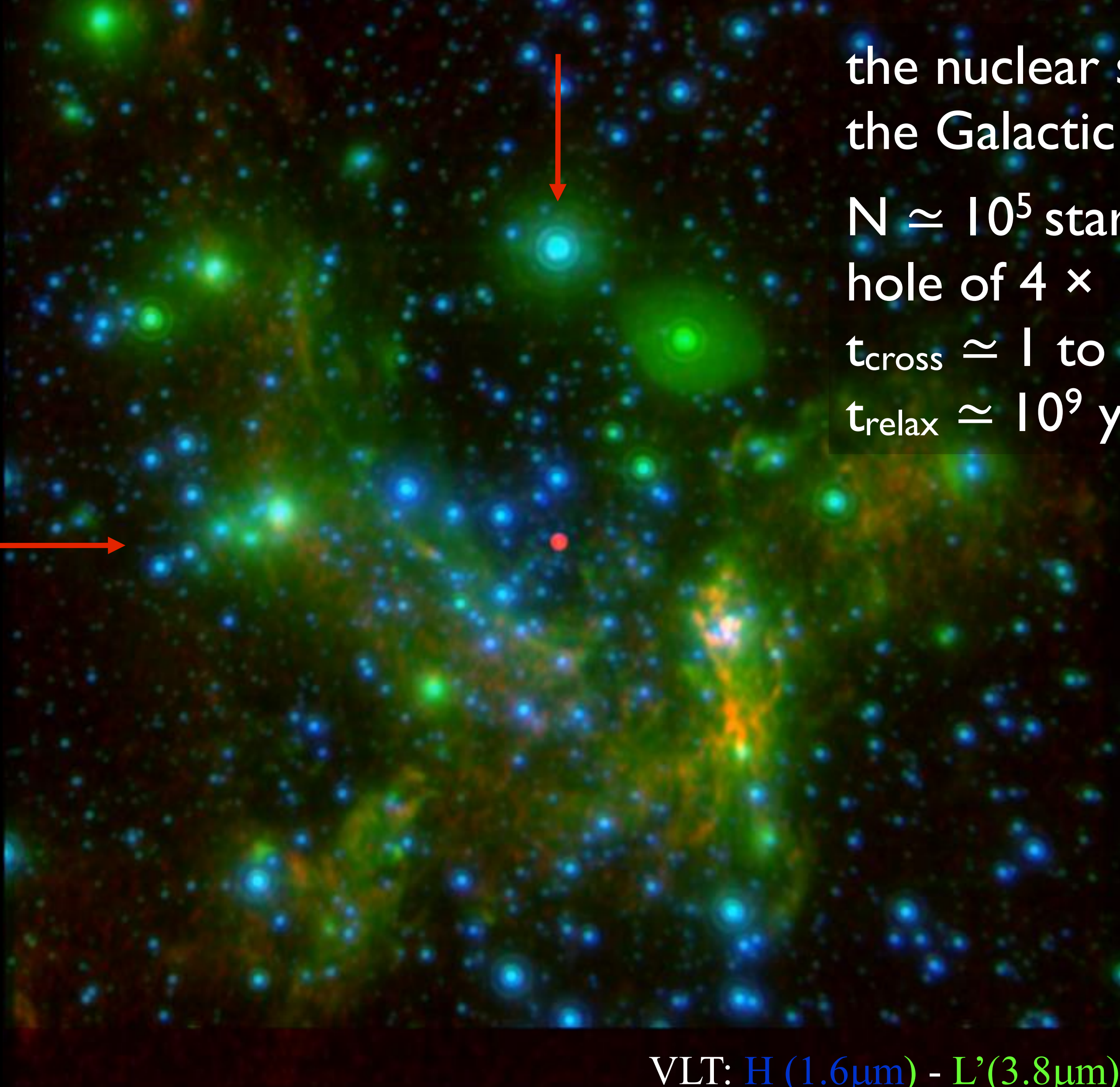
$t_{relax} \simeq 10^9$ yr

Genzel (2015)



10”
(0.4 pc)

VLT: H (1.6μm) - L’(3.8μm)
VLA: 1.3cm



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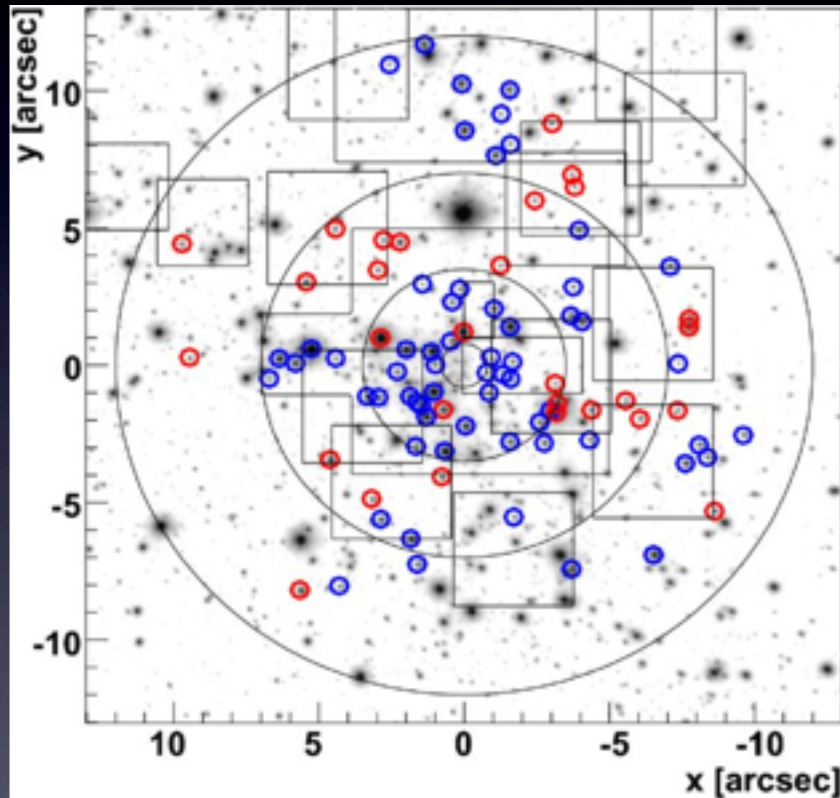
VLA: 1.3cm

with Bence Kocsis (IAS and Eötvös University)

The stellar disk(s) in the Galactic center

1 pc = 25”

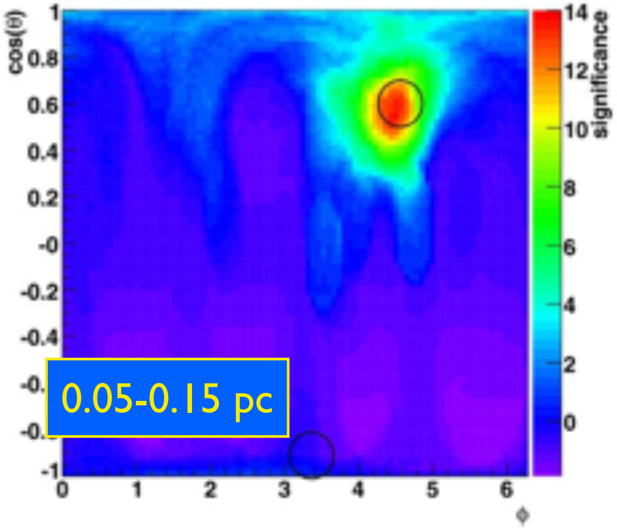
- ~ 100 massive young stars found in the central parsec
- age 6 Myr; implied star-formation rate is so high that it must be episodic
- line-of-sight velocities measured by Doppler shift and angular velocities measured by astrometry (five of six phase-space coordinates)
- velocity vectors lie close to a plane, implying that many of the stars are in a disk (Levin & Beloborodov 2003)
- there is strong evidence for a second disk co-spatial with the first but roughly perpendicular to it

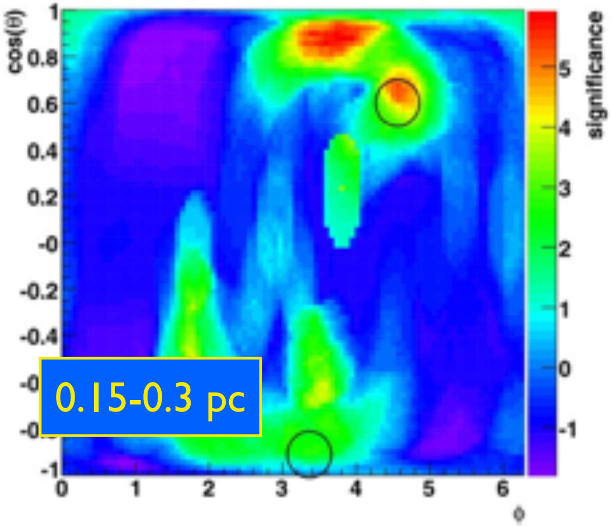


Bartko et al. (2009)

blue = clockwise rotation (61 stars)

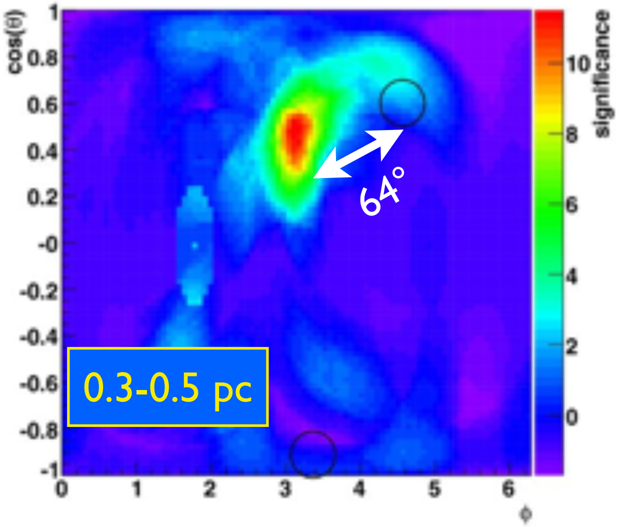
red = counter-clockwise rotation (29 stars)





- plots show probability distribution of orbit normals of the young stars

1 pc = 25''

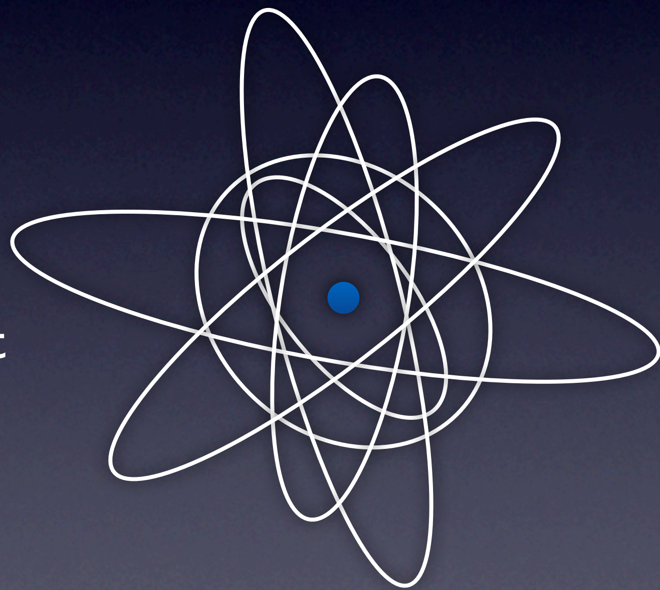


Bartko et al. (2009)

- **clockwise disk:**
  - warped (best-fit normals in inner and outer image differ by 64°)
  - disk is less well-formed at larger radii
- **counter-clockwise disk:**
  - weaker evidence
  - localized between 0.1 and 0.3 pc
- disks are embedded in a spherical cluster of old, fainter stars with M(0.1 pc) ~ 1×10^5 $M_\odot$ compared to $M_\bullet = 4\times10^6\ M_\odot$

Resonant relaxation

- inside ~0.5 pc gravitational field is dominated by the black hole ($M_{stars} < 10^5\ M_\odot$, $M_\bullet \sim 4\times10^6\ M_\odot$) and therefore is nearly spherical
- on timescales longer than the apsidal precession period each stellar orbit can be thought of as a disk or annulus
- each disk exerts a torque on all other disks
- mutual torques can lead to relaxation of orbit normals or angular momenta
- energy (semi-major axis) and scalar angular momentum (or eccentricity) of each orbit is conserved, but orbit normal is not



Rauch & Tremaine (1996)

Resonant relaxation

Interaction energy between stars i and j is $m_i m_j f(a_i, a_j, e_i, e_j, \cos\,\mu_{ij})$ where μ_{ij} is the angle between the orbit normals

eccentricities

masses

semi-major axes

Toy model:

Simplify this drastically by assuming equal masses, equal semi-major axes, circular orbits, and neglecting all harmonics other than quadrupole

Resulting interaction energy between two stars i and j is just

$$- C \cos^2 \mu_{ij}$$

where $\boldsymbol{\mu}_{ij}$ is the angle between the two orbit normals $\mathbf{n}_i$ and $\mathbf{n}_j$

$$\frac{d\hat{\mathbf{n}}_i}{dt} = -\frac{2C}{\sqrt{GM_\bullet a}} \sum_{j \neq i} (\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)\, \hat{\mathbf{n}}_i \times \hat{\mathbf{n}}_j$$

Maier-Saupe model

Resonant relaxation

Interaction energy between two stars is

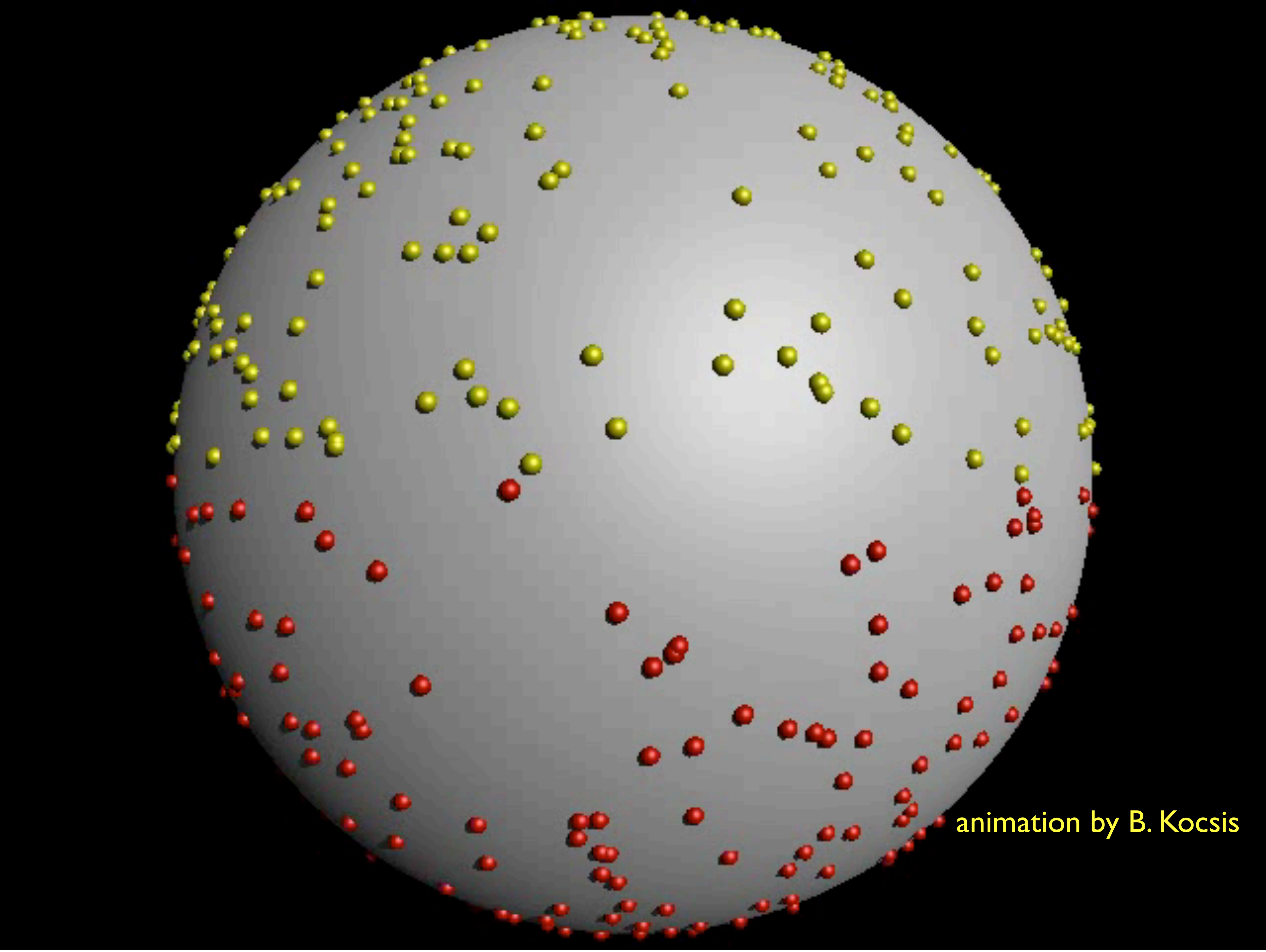
H = -C cos² μ

where **μ** is the angle between the two orbit normals

- 800 stars
- each point represents tip of orbit normal
- orbit normals initially in northern hemisphere are yellow, south is red

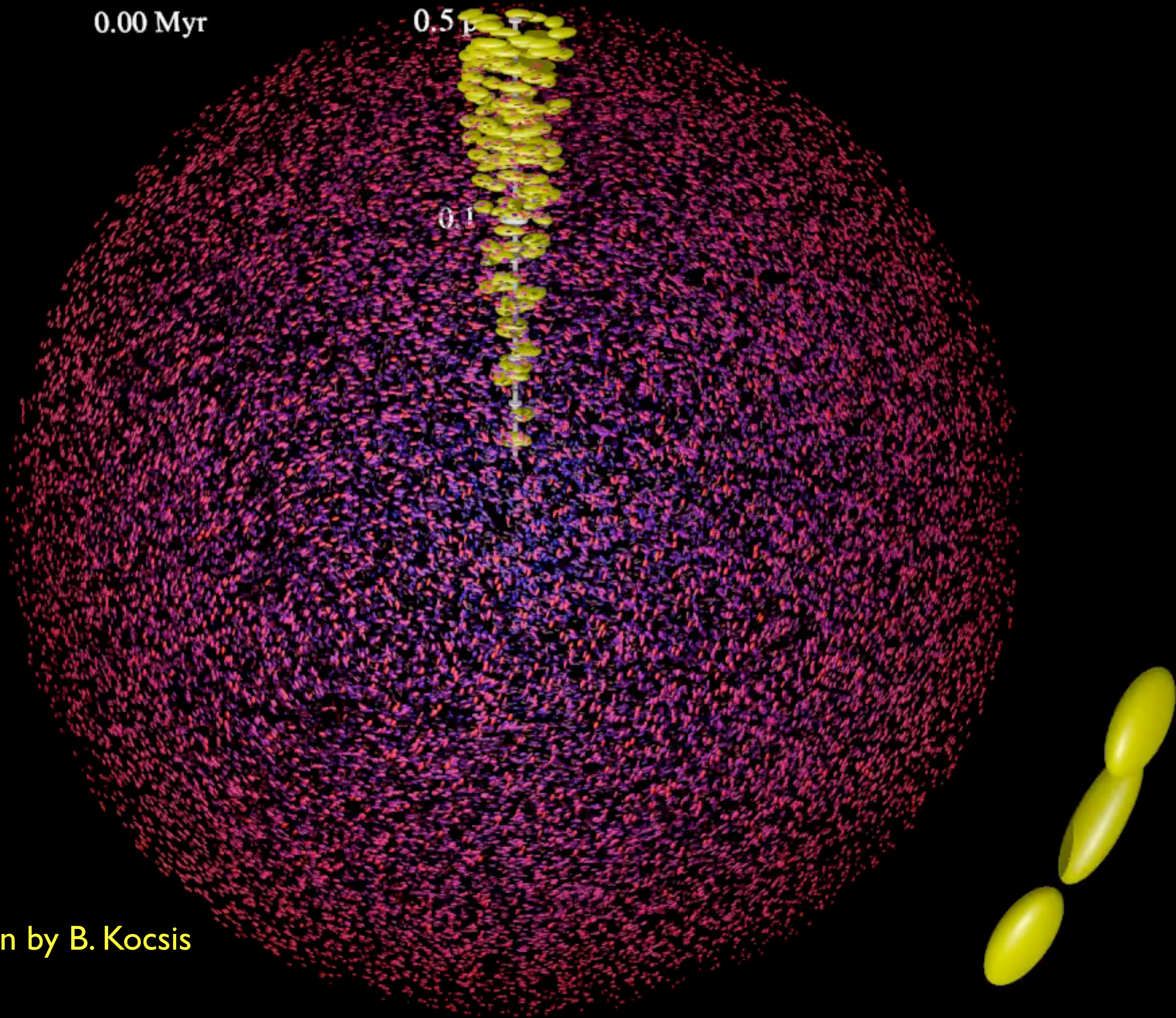
$$\frac{d\mathbf{n}_i}{dt} = -\frac{2C}{\sqrt{GMa}}\sum_{j\neq i}(\mathbf{n}_i\cdot\mathbf{n}_j)\mathbf{n}_i\times\mathbf{n}_j$$

animation by B. Kocsis

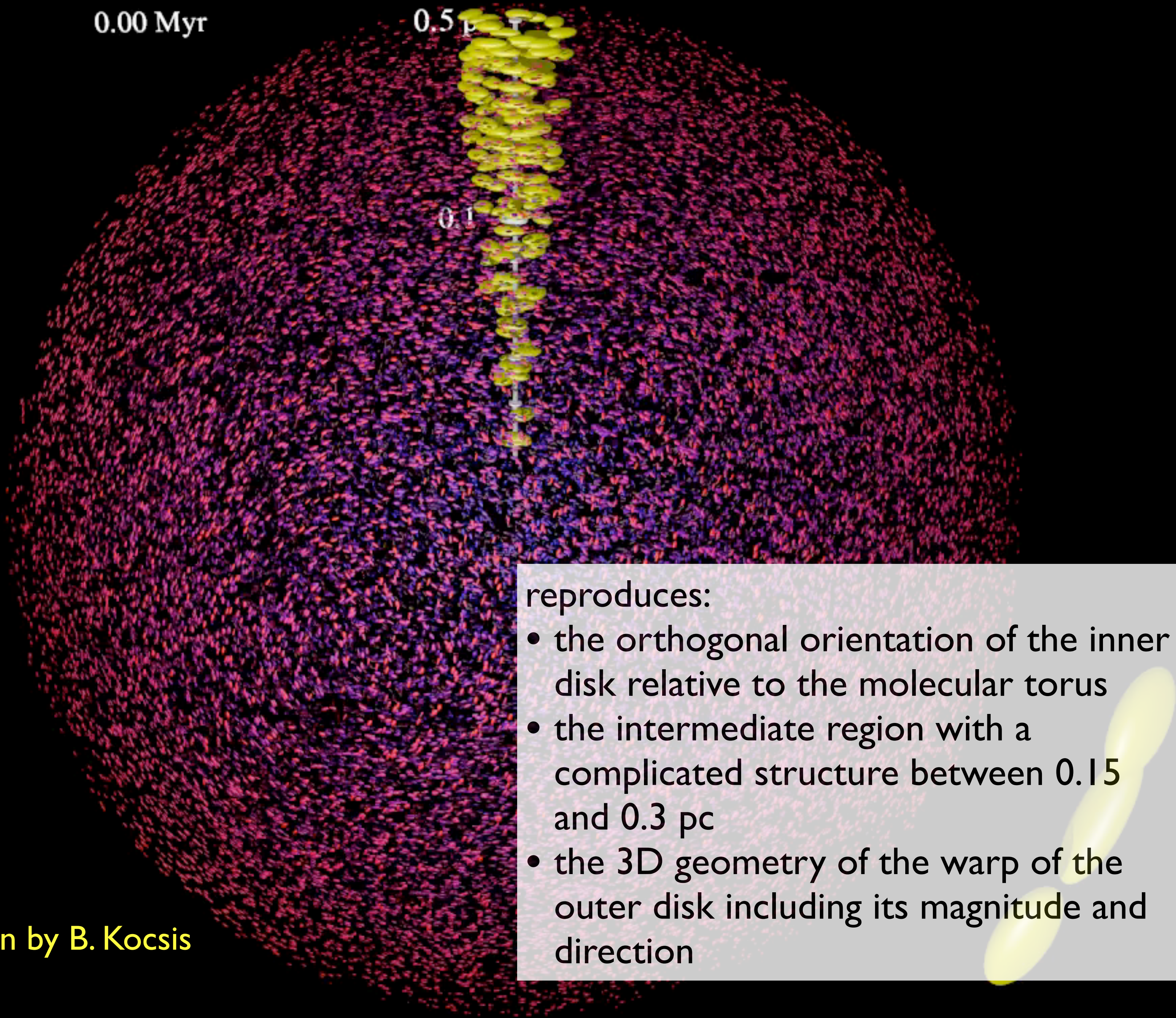

animation by B. Kocsis

- integrate orbit-averaged equations of motion
- yellow = disk stars, blue-red = stars in spherical cluster, colored by increasing radius
- large yellow = molecular torus
- direction and radius of each point represents direction of angular-momentum vector and semi-major axis of star
- 32K stars
- each point represents tip of orbit normal

animation by B. Kocsis



animation by B. Kocsis



reproduces:

- the orthogonal orientation of the inner disk relative to the molecular torus
- the intermediate region with a complicated structure between 0.15 and 0.3 pc
- the 3D geometry of the warp of the outer disk including its magnitude and direction

animation by B. Kocsis

"All models are wrong, but some are useful"

Box & Draper (1987)